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Asymptotic Properties of Estimators for the Linear Panel Regression Model with Individual Effects and Serially Correlated Errors: The Case of Stationary and Non-Stationary Regressors and Residuals

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Abstract

This paper studies the asymptotic properties of standard panel data estimators in a simple panel regression model with error component disturbances. Both the regressor and the remainder disturbance term are assumed to be autoregressive and possibly non-stationary. Asymptotic distributions are derived for the standard panel data estimators including ordinary least squares, fixed effects, first-difference, and generalized least squares (GLS) estimators when both T and n are large. We show that all the estimators have asymptotic normal distributions and have different convergence rates dependent on the non-stationarity of the regressors and the remainder disturbances. We show using Monte Carlo experiments that the loss in efficiency of the OLS, FE and FD estimators relative to true GLS can be substantial.

Key Words: Panel Data, OLS, Fixed-Effects, First-Difference, GLS
JEL classification: C33

Asymptotic Properties of Estimators for the Linear Panel Regression Model with Individual effects and Serially Correlated Errors: The Case of Stationary and Non-Stationary Regressors and Residuals

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April 22, 2007

Abstract

This paper studies the asymptotic properties of standard panel data estimators in a simple panel regression model with error component disturbances. Both the regressor and the remainder disturbance term are assumed to be autoregressive and possibly non-stationary. Asymptotic distributions are derived for the standard panel data estimators including ordinary least squares, fixed effects, first-difference, and generalized least squares (GLS) estimators when both T and n are large. We show that all the estimators have asymptotic normal distributions and have different convergence rates dependent on the non-stationarity of the regressors and the remainder disturbances. We show using Monte Carlo experiments that the loss in efficiency of the OLS, FE and FD estimators relative to true GLS can be substantial.

Key Words: *Panel Data, OLS, Fixed-Effects, First-Difference, GLS.*

1 Introduction

Econometricians have long been concerned with conditions under which the ordinary least squares (OLS) estimator is asymptotically efficient. The standard textbook result is that, under a general variance-covariance

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structure on the disturbances, the OLS estimator is less efficient than generalized least squares (GLS). This is well documented for the case of stationary autoregressive disturbances and stationary regressors. However, Phillips and Park (1988) showed that in a regression with integrated regressors, OLS and GLS are asymptotically equivalent.

Recently, Choi (1999) studied the limiting distributions of the fixed effects (FE), GLS, and within-GLS estimators for a panel data regression model with autoregressive disturbances, while Choi (2002) extended this work to instrumental variables (IV) estimation. Phillips and Moon (1999) presented a fundamental framework for studying sequential and joint limit theories in nonstationary panel data analysis, while Kao (1999) studied the asymptotic properties of the FE estimator of a spurious regression and proposed residual-based tests for panel cointegration. See Baltagi and Kao (2000), Choi (2006) and Breitung and Pesaran (2006) for recent surveys of this rapidly growing subject. In an early finding, Baltagi and Krämer (1997) showed the equivalence of the GLS and FE estimators in a simple panel data regression with time trend as a regressor. Kao and Emerson (2004a, 2004b) extended Baltagi and Krämer to a model with serially correlated remainder errors. Kao and Emerson showed that the FE estimator is asymptotically equivalent to GLS when the error term is $I(0)$; but that GLS is more efficient than FE when the error term is $I(1)$. It is known that the panel time trend can be seen as a special case of the panel regression with a non-zero drift $I(1)$ regressor.

This paper extends the literature by studying the asymptotic properties of OLS, FE, first difference (FD) and GLS in the panel regression with an autocorrelated regressor and an autocorrelated remainder error (both of which can be stationary or nonstationary). We show that when the error term is $I(0)$ and the regressor is $I(1)$, the FE estimator is asymptotically equivalent to the GLS estimator and OLS is less efficient than GLS (due to a slower convergence speed). However, when the error term and the regressor are $I(1)$, GLS is more efficient than the FE estimator since GLS is $\sqrt{nT}$ consistent, while FE is $\sqrt{n}$ consistent. This implies that GLS is the preferred estimator under both cases (i.e., regression error is either $I(0)$ or $I(1)$).

All asymptotic results in this paper assume that $T \to \infty$ followed by $n \to \infty$. We use $(n, T) \overset{\text{seq}}{\to} \infty$ to denote this sequential limit. We write the integral $\int_0^1 W(s)ds$ as $\int W$ and $\tilde{W}$ as $W - \int W$ when there is no ambiguity over limits. $\Rightarrow$ to denote weak convergence, $\equiv$ to denote equivalence in distribution, $\overset{p}{\to}$ to denote convergence in probability, $[x]$ to denote the largest integer $\leq x$, $I(0)$ and $I(1)$ to signify a time series that is integrated of order zero and one, respectively, and $BM(\Omega)$ to denote Brownian motion with covariance matrix Ω. All proofs are collected in an appendix available upon request from the authors.

2 The Model and Assumptions

Consider the following panel regression:

$$y_{it} = \alpha + x_{it}\beta + u_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T \tag{1}$$

where $u_{it} = \mu_i + \nu_{it}$, and α and β are scalars. For simplicity, we consider the case of one regressor, but our results can be extended to the multiple regressor case. We assume that $\mu_i \sim iid(0, \sigma_\mu^2)$ and $\{\nu_{it}\}$ is AR(1)

$$\nu_{it} = \rho\nu_{it-1} + e_{it}, \qquad |\rho| \leq 1 \tag{2}$$

where e_{it} is a white noise process with variance ϖ_e^2.

Let x_{it} be also an AR(1) such that

$$x_{it} = \lambda x_{it-1} + \varepsilon_{it}, \qquad |\lambda| \leq 1 \tag{3}$$

where ε_{it} is a white noise process with variance ϖ_ε^2. In this paper, we assume that

$$E(\mu_i | x_{it}) = 0. \tag{4}$$

The initialization of this system is $y_{i1} = x_{i1} = O_p(1)$ for all i. Our interest is in the estimation of the common slope β. This paper shows that the asymptotic properties of OLS, FE, FD, and GLS estimators depend crucially on the serial correlation properties of x_{it} and v_{it}. When y_{it} and x_{it} are both $I(1)$ but v_{it} is $I(0)$, equation (1) is a panel cointegrated model. On the other hand, when v_{it} is $I(1)$ and y_{it} and x_{it} are both $I(1)$, equation (1) is a panel spurious model. FE estimators for panel cointegrated and panel spurious models have been discussed in Phillips and Moon (1999) and Kao (1999). The case of a panel time trend model, $x_{it} = t$, has been studied by Baltagi and Krämer (1997) and Kao and Emerson (2004a, 2004b).

Next, we characterize the innovation vector $\mathbf{w}_{it} = (e_{it}, \varepsilon_{it})'$. We assume that $\mathbf{w}_{it}$ is a linear process that satisfies the following assumption:

Assumption 1 *For each i, we assume:*

1. $\mathbf{w}_{it} = \Pi(L)\boldsymbol{\eta}_{it} = \sum_{j=0}^{\infty} \Pi_j \boldsymbol{\eta}_{it-j}, \sum_{j=0}^{\infty} j^a \|\Pi_j\| < \infty, |\Pi(1)| \neq 0$ for some $a > 1$.
2. For a given i, $\boldsymbol{\eta}_{it}$ is i.i.d. with zero mean and variance-covariance matrix Ξ, and finite fourth order cumulants.

Assumption 2 *We assume $\boldsymbol{\eta}_{it}$ and $\boldsymbol{\eta}_{jt}$ are independent for $i \neq j$. That is we assume cross-sectional independence for our model.*

Assumption 1 implies that the partial sum process $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}\mathbf{w}_{it}$ satisfies the following multivariate invariance principle:

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}\mathbf{w}_{it} \Rightarrow \mathbf{B}_i(r) = \mathbf{BM}_i\left(\Omega\right) \text{ as } T\to\infty \text{ for all } i, \tag{5}$$

where

$$\mathbf{B}_i = \begin{bmatrix} B_{ei} \\ B_{\varepsilon i}\end{bmatrix}.$$

The long-run 2×2 covariance matrix of $\{\mathbf{w}_{it}\}$ is given by

$$\begin{aligned}\Omega &= \sum_{j=-\infty}^{\infty} E\left(\mathbf{w}_{ij}\mathbf{w}_{i0}^{'}\right) \\ &= \Pi(1)\Xi\Pi(1)^{'} \\ &= \begin{bmatrix} \varpi_e^2 & \varpi_{e\varepsilon} \\ \varpi_{e\varepsilon} & \varpi_\varepsilon^2 \end{bmatrix}.\end{aligned}$$

The long-run covariance matrix can be decomposed into $\Omega = \Sigma + 2\Gamma$, where

$$\Gamma = \sum_{j=1}^{\infty} E\left(\mathbf{w}_{ij}\mathbf{w}_{i0}^{'}\right) = \begin{bmatrix} \gamma_e^2 & \gamma_{e\varepsilon} \\ \gamma_{e\varepsilon} & \gamma_\varepsilon^2 \end{bmatrix} \tag{6}$$

and

$$\Sigma = E\left(\mathbf{w}_{i0}\mathbf{w}_{i0}^{'}\right) = \begin{bmatrix} \sigma_e^2 & \sigma_{e\varepsilon} \\ \sigma_{e\varepsilon} & \sigma_\varepsilon^2 \end{bmatrix}. \tag{7}$$

Assuming ϖ_ε^2 is non-zero, we define

$$\varpi_{e.\varepsilon} = \varpi_e^2 - \frac{\varpi_{e\varepsilon}^2}{\varpi_\varepsilon^2}. \tag{8}$$

Then, B_i can be rewritten as

$$\mathbf{B}_i = \begin{bmatrix} B_{ei} \\ B_{\varepsilon i}\end{bmatrix} = \begin{bmatrix} \varpi_{e.\varepsilon} & \varpi_{e\varepsilon}/\varpi_\varepsilon \\ 0 & \varpi_\varepsilon \end{bmatrix}\begin{bmatrix} V_i \\ W_i \end{bmatrix} \tag{9}$$

where $\begin{bmatrix} V_i \\ W_i \end{bmatrix} = \mathbf{BM}\left(I\right)$ is a standardized Brownian motion. Define the one-sided long-run covariance

$$\begin{aligned}\Delta &= \Sigma + \Gamma \\ &= \sum_{j=0}^{\infty} E\left(\mathbf{w}_{ij}\mathbf{w}_{i0}^{'}\right)\end{aligned}$$

with

$$\Delta = \begin{bmatrix} \delta_e^2 & \delta_{e\varepsilon} \\ \delta_{e\varepsilon} & \delta_\varepsilon^2 \end{bmatrix}.$$

The assumption of constant variances/covariances across i, such as in Ω, Σ, and Γ is used to simplify the notation. It can be extended into the case where different variances are allowed for different i at the expense of more complicated notation.

3 OLS Estimator

The OLS estimator of β is given by

$$\widehat{\beta}_{OLS} = \frac{\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(y_{it}-\overline{\overline{y}}\right)}{\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^{2}} \tag{10}$$

where $\overline{\overline{x}} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}$ and $\overline{\overline{y}} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}y_{it}$.

Theorem 1 *Under Assumptions* $1-2$*, we obtain the following results:*

1. If $|\rho| < 1$ and $|\lambda| < 1$,

   (a) $\widehat{\beta}_{OLS} - \beta \xrightarrow{p} \frac{1-\lambda^{2}}{\sigma_{\varepsilon}^{2}}\left[\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\right]$,

   (b) $\sqrt{nT}\left(\widehat{\beta}_{OLS} - \beta - \tau_{1NT}^{OLS}\right) \Rightarrow N\left(0, \kappa_{1}^{OLS}\right)$,

   where $\tau_{1NT}^{OLS} = \frac{\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E(x_{it}\nu_{it})}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^{2}}$, $\kappa_{1}^{OLS} = \frac{1}{\sigma_{\varepsilon}^{4}}\left[\sigma_{\mu}^{2}\varpi_{\varepsilon}^{2} + \frac{\left(1-\lambda^{2}\right)^{2}}{(1-\rho\lambda)^{2}}\left[\psi_{00} + \sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r} + \sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}\right]\right]$,
   $\psi_{0r} = E\left(\varepsilon_{i(t-r)}^{2}e_{it}^{2}\right)$, $\psi_{r0} = E\left(\varepsilon_{it}^{2}e_{i(t-r)}^{2}\right)$, and $\psi_{00} = E\left(\varepsilon_{it}^{2}e_{it}^{2}\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,

   (a) $\widehat{\beta}_{OLS} - \beta \xrightarrow{p} \frac{(1+\lambda)\left(-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}\right)}{\sigma_{\varepsilon}^{2}}$

   (b) $\sqrt{n}\left(\widehat{\beta}_{OLS} - \beta - \tau_{2nT}^{OLS}\right) \Rightarrow N\left(0, \kappa_{2}^{OLS}\right)$,

   where $\tau_{2nT}^{OLS} = \frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\nu_{i(t-1)}e_{it}\right)\frac{\varpi_{e\varepsilon}}{\varpi_{e}^{2}}+\delta_{e\varepsilon}}{(1-\lambda)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^{2}}$, $\kappa_{2}^{OLS} = \frac{(1+\lambda)^{2}\varpi_{\varepsilon.e}\varpi_{e}^{2}}{2\sigma_{\varepsilon}^{4}}$.

3. If $|\rho| < 1$ and $\lambda = 1$,

   (a) $\sqrt{T}\left(\widehat{\beta}_{OLS} - \beta\right) \xrightarrow{p} 0$,

   (b) $\sqrt{nT}\left(\widehat{\beta}_{OLS} - \beta\right) \Rightarrow N\left(0, \kappa_{3}^{OLS}\right)$,

   where $\kappa_{3}^{OLS} = \frac{4\sigma_{\mu}^{2}}{3\varpi_{\varepsilon}^{2}}$.

4. If $\rho = 1$ and $\lambda = 1$,

(a) $\widehat{\beta}_{OLS} - \beta \xrightarrow{p} \frac{2\delta_{\varepsilon e}}{\varpi_{\varepsilon}^2}$,

(b) $\sqrt{n}\left(\widehat{\beta}_{OLS} - \beta - \tau_{4NT}^{OLS}\right) \Rightarrow N\left(0, \kappa_4^{OLS}\right)$,

where $\tau_{4nT}^{OLS} = \frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T} x_{i(t-1)}\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\delta_{\varepsilon e}}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)^2}$, $\kappa_4^{OLS} = \frac{2\varpi_{e.\varepsilon}}{3\varpi_{\varepsilon}^2}$.

It is important to note that $\varpi_{e.\varepsilon}/\varpi_{\varepsilon}^2$ can be seen as the long-run signal-to-noise ratio. The OLS estimator ignores the individual effects in the disturbance term. Thus, the variance of μ_i, i.e., σ_μ^2 might appear in the variance-covariance matrix of $\widehat{\beta}_{OLS}$ depending on the case considered. In case 1, both μ_i and ν_{it} affect the variance of $\widehat{\beta}_{OLS}$. In cases 2 and 4, ν_{it} dominates μ_i. In case 3, μ_i dominates ν_{it} and hence the convergence speed is $\sqrt{nT}$, which differs from the T-asymptotics in the panel cointegration literature. Also the asymptotic normality of the OLS estimator comes naturally. When summing across i, the nonstandard asymptotic distribution due to unit root in the time dimension, such as for cases 2-4, is smoothed out.

Corollary 1 *When $E\left(e_{it}\varepsilon_{i(t+k)}\right) = 0$ for all i and k, under the assumptions in Theorem 1, then*

1. If $|\rho| < 1$ and $|\lambda| < 1$,

   $\sqrt{nT}\left(\widehat{\beta}_{OLS} - \beta\right) \Rightarrow N\left(0, \frac{\sigma_\mu^2}{\sigma_\varepsilon^2} + \frac{(1-\lambda^2)^2\left[\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}\right]}{(1-\rho\lambda)^2\sigma_\varepsilon^4}\right)$.

   When ε_{it} and e_{it} are independent, $\sqrt{nT}\left(\widehat{\beta}_{OLS} - \beta\right) \Rightarrow N\left(0, \frac{\sigma_\mu^2}{\sigma_\varepsilon^2} + \frac{(1+\rho\lambda)(1-\lambda^2)\sigma_e^2}{(1-\rho\lambda)(1-\rho^2)\sigma_\varepsilon^2}\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,

   $\sqrt{n}\left(\widehat{\beta}_{OLS} - \beta\right) \Rightarrow N\left(0, \frac{(1-\lambda)^2\sigma_e^2}{2\sigma_\varepsilon^2}\right)$,

3. If $|\rho| < 1$ and $\lambda = 1$,

   $\sqrt{nT}\left(\widehat{\beta}_{OLS} - \beta\right) \Rightarrow N\left(0, \frac{4\sigma_\mu^2}{3\sigma_\varepsilon^2}\right)$,

4. If $\rho = 1$ and $\lambda = 1$,

   $\sqrt{n}\left(\widehat{\beta}_{OLS} - \beta\right) \Rightarrow N\left(0, \frac{2\sigma_e^2}{3\sigma_\varepsilon^2}\right)$.

Corollary 1 follows directly from Theorem 1.

4 FE Estimator

The Fixed-Effects estimator of β is given by

$$\widehat{\beta}_{FE} = \frac{\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{x}_i\right)\left(y_{it} - \overline{y}_i\right)}{\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{x}_i\right)^2}, \tag{11}$$

where $\overline{x}_i = \frac{1}{T}\sum_{t=1}^{T} x_{it}$ and $\overline{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}$.

Theorem 2 *Under Assumptions* $1-2$*, we have the following results:*

1. If $|\rho|<1$ and $|\lambda|<1$,

   (a) $\widehat{\beta}_{FE}-\beta\overset{p}{\to}\frac{1-\lambda^2}{\sigma_\varepsilon^2}\left[\lim\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T E\left(x_{it}\nu_{it}\right)\right]$,

   (b) $\sqrt{nT}\left(\widehat{\beta}_{FE}-\beta-\tau_{1nT}^{FE}\right)\Rightarrow N\left(0,\kappa_1^{FE}\right)$,

   where $\tau_{1nT}^{FE}=\frac{\lim\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T E(x_{it}\nu_{it})}{\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T(x_{it}-\overline{x}_i)^2}$, $\kappa_1^{FE}=\frac{\left(1-\lambda^2\right)^2\left[\psi_{00}+\sum_{r=1}^\infty\lambda^{2r}\psi_{0r}+\sum_{r=1}^\infty\rho^{2r}\psi_{r0}\right]}{(1-\rho\lambda)^2\sigma_\varepsilon^4}$, $\psi_{0r}=E\left(\varepsilon_{i(t-r)}^2e_{it}^2\right)$, $\psi_{r0}=E\left(\varepsilon_{it}^2e_{i(t-r)}^2\right)$, and $\psi_{00}=E\left(\varepsilon_{it}^2e_{it}^2\right)$.

2. If $\rho=1$ and $|\lambda|<1$,

   (a) $\widehat{\beta}_{FE}-\beta\overset{p}{\to}\frac{(1+\lambda)\left(-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}\right)}{\sigma_\varepsilon^2}$,

   (b) $\sqrt{n}\left(\widehat{\beta}_{FE}-\beta-\tau_{2nT}^{FE}\right)\Rightarrow N\left(0,\kappa_2^{FE}\right)$,

   where $\tau_{2nT}^{FE}=\frac{\left(\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T(\nu_{it}-\bar{\nu}_i)e_{it}\right)\frac{\varpi_{e\varepsilon}}{\varpi_e^2}+\delta_{e\varepsilon}}{(1-\lambda)\frac{1}{n}\sum_{i=1}^n\frac{1}{T^2}\sum_{t=1}^T(x_{it}-\overline{x}_i)^2}$ and $\kappa_2^{FE}=\frac{(1-\lambda)^2\varpi_{\varepsilon.e}\varpi_e^2}{6\sigma_\varepsilon^4}$.

3. If $|\rho|<1$ and $\lambda=1$,

   (a) $T\left(\widehat{\beta}_{FE}-\beta\right)\overset{p}{\to}\frac{-3\varpi_{\varepsilon e}+6\delta_{\varepsilon e}}{(1-\rho)\varpi_\varepsilon^2}$,

   (b) $\sqrt{n}T\left(\widehat{\beta}_{FE}-\beta\right)-\sqrt{n}\tau_{3nT}^{FE}\Rightarrow N\left(0,\kappa_3^{FE}\right)$,

   where $\tau_{3nT}^{FE}=\frac{\left(\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T(x_{it}-\overline{x}_i)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\delta_{\varepsilon e}}{(1-\rho)\frac{1}{n}\sum_{i=1}^n\frac{1}{T^2}\sum_{t=1}^T(x_{it}-\overline{x}_i)^2}$ and $\kappa_3^{FE}=\frac{6\varpi_{e.\varepsilon}}{(1-\rho)^2\varpi_\varepsilon^2}$.

4. If $\rho=1$ and $\lambda=1$,

   (a) $\widehat{\beta}_{FE}-\beta\overset{p}{\to}\frac{\varpi_{\varepsilon e}+6\delta_{\varepsilon e}}{\varpi_\varepsilon^2}$,

   (b) $\sqrt{n}\left(\widehat{\beta}_{FE}-\beta-\tau_{4nT}^{FE}\right)\Rightarrow N\left(0,\kappa_4^{FE}\right)$,

   where $\tau_{4nT}^{FE}=\frac{\left(\frac{1}{n}\sum_{i=1}^n\frac{1}{T^2}\sum_{t=1}^T(x_{it}-\overline{x}_i)^2\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\delta_{\varepsilon e}}{\frac{1}{n}\sum_{i=1}^n\frac{1}{T^2}\sum_{t=1}^T(x_{it}-\overline{x}_i)^2}$ and $\kappa_4^{FE}=\frac{2\varpi_{e.\varepsilon}}{5\varpi_\varepsilon^2}$.

Note $\varpi_{\varepsilon e}$ is due to the endogeneity of the regressor x_{it}, and $\delta_{\varepsilon e}$ is due to serial correlation. Because $u_{it}-\overline{u}_i=\nu_{it}-\overline{\nu}_i$, the individual effect μ_i is eliminated for each individual.

Corollary 2 *When* $E\left(e_{it}\varepsilon_{i(t+k)}\right)=0$ *for all* i *and* k*, under the same conditions as for Theorem 2, then*

1. If $|\rho|<1$ and $|\lambda|<1$,

   $\sqrt{nT}\left(\widehat{\beta}_{FE}-\beta\right)\Rightarrow N\left(0,\frac{\left(1-\lambda^2\right)^2\left[\psi_{00}+\sum_{r=1}^\infty\lambda^{2r}\psi_{0r}+\sum_{r=1}^\infty\rho^{2r}\psi_{r0}\right]}{(1-\rho\lambda)^2\sigma_\varepsilon^4}\right)$.

   If ε_{it} and e_{it} are independent, $\sqrt{nT}\left(\widehat{\beta}_{FE}-\beta\right)\Rightarrow N\left(0,\frac{(1+\rho\lambda)\left(1-\lambda^2\right)\sigma_e^2}{(1-\rho\lambda)(1-\rho^2)\sigma_\varepsilon^2}\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,

   $\sqrt{n}\left(\widehat{\beta}_{FE} - \beta\right) \Rightarrow N\left(0, \frac{(1-\lambda)^2\sigma_e^2}{6\sigma_\varepsilon^2}\right).$

3. If $|\rho| < 1$ and $\lambda = 1$,

   $\sqrt{n}T\left(\widehat{\beta}_{FE} - \beta\right) \Rightarrow N\left(0, \frac{6\sigma_e^2}{(1-\rho)^2\sigma_\varepsilon^2}\right).$

4. If $\rho = 1$ and $\lambda = 1$,

   $\sqrt{n}\left(\widehat{\beta}_{FE} - \beta\right) \Rightarrow N\left(0, \frac{2\sigma_e^2}{5\sigma_\varepsilon^2}\right).$

Corollary 2 follows directly from Theorem 2. Note that case 1 is the textbook result under the assumptions of stationarity of the regressor and the disturbance term. Case 2 is new. Case 3 is discussed by Phillips and Moon (1999) and Kao and Chiang (2000). Case 4 is discussed in Kao (1999).

5 FD Estimator

The First-difference estimator of β is given by

$$\widehat{\beta}_{FD} = \frac{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})(y_{it} - y_{it-1})}{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2}. \tag{12}$$

Theorem 3 *Under Assumptions* $1-2$, *we obtain the following results:*

1. If $|\rho| < 1$ and $|\lambda| < 1$,

   (a) $\widehat{\beta}_{FD} - \beta \xrightarrow{p} \frac{\lim \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T E[(x_{it}-x_{it-1})(\nu_{it}-\nu_{it-1})]}{\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda}},$

   (b) $\sqrt{nT}\left(\widehat{\beta}_{FD} - \beta - \tau_{1nT}^{FD}\right) \Rightarrow N\left(0, \kappa_1^{FD}\right)$, where

   $$\tau_{1nT}^{FD} = \frac{\lim \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T E\left[(x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1})\right]}{\frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2},$$

   and

   $$\kappa_1^{FD} = \frac{(2-\lambda-\rho)^2\psi_{00} + \sum_{r=1}^\infty \left(-\rho^{r-1} + 2\rho^r - \rho^{r+1}\right)^2 \psi_{0r} + \sum_{r=1}^\infty \left(-\lambda^{r-1} + 2\lambda^r - \lambda^{r+1}\right)^2 \psi_{r0}}{(1-\rho\lambda)^2\left(\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda}\right)^2}.$$

2. If $\rho = 1$ and $|\lambda| < 1$,

   (a) $\widehat{\beta}_{FD} - \beta \xrightarrow{p} \frac{\lim \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T E[(\varepsilon_{it}+(\lambda-1)x_{it-1})e_{it}]}{\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda}},$

(b) $\sqrt{nT}\left(\widehat{\beta}_{FE}-\beta-\tau_{2nT}^{FD}\right)\Rightarrow N\left(0,\kappa_2^{FD}\right)$, where

$$\tau_{2nT}^{FD}=\frac{\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left[\left(\varepsilon_{it}+(\lambda-1)\,x_{it-1}\right)e_{it}\right]}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2},$$

and

$$\kappa_2^{FD}=\frac{2\psi_{00}}{(1+\lambda)\left(\frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda}\right)^2}.$$

3. If $|\rho|<1$ and $\lambda=1$,

   (a) $\widehat{\beta}_{FD}-\beta\xrightarrow{p}\frac{1}{\sigma_\varepsilon^2}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left[\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)\right]$,

   (b) $\sqrt{nT}\left(\widehat{\beta}_{FD}-\beta-\tau_{3nT}^{FD}\right)\Rightarrow N\left(0,\Pi_3^{FD}\right)$,
   where $\tau_{3nT}^{FD}=\frac{\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E[\varepsilon_{it}(\nu_{it}-\nu_{it-1})]}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-x_{it-1})^2}$ and $\kappa_3^{FD}=\frac{2\psi_{00}}{(1+\rho)\sigma_\varepsilon^4}$.

4. If $\rho=1$ and $\lambda=1$,

   (a) $\widehat{\beta}_{FD}-\beta\xrightarrow{p}\frac{\sigma_{\varepsilon e}}{\sigma_\varepsilon^2}$,

   (b) $\sqrt{nT}\left(\widehat{\beta}_{FD}-\beta-\tau_{4nT}^{FD}\right)\Rightarrow N\left(0,\kappa_4^{FD}\right)$,
   where $\tau_{4nT}^{FD}=\frac{\sigma_{\varepsilon e}}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-x_{it-1})^2}$ and $\kappa_4^{FD}=\frac{\varpi_{e.\varepsilon}\varpi_\varepsilon^2}{\sigma_\varepsilon^4}$.

Similar to the FE estimator, the individual effect μ_i is also eliminated by the FD estimator because $u_{it}-u_{it-1}=\nu_{it}-\nu_{it-1}$. In cases 2 and 4, $\rho=1$, and the FD estimator is asymptotically equivalent to the GLS estimator because both methods transform the disturbance from $I(1)$ into $I(0)$. Actually, the FD estimator is mathematically the same as the GLS estimator except for the omission of the first observation for each individual.

Corollary 3 *When $E\left(e_{it}\varepsilon_{i(t+k)}\right)=0$ for all i and k, under the same conditions as for Theorem 3, then*

1. If $|\rho|<1$ and $|\lambda|<1$,
   $\sqrt{nT}\left(\widehat{\beta}_{FD}-\beta\right)\Rightarrow N(0,\frac{(1+\lambda)^2\left[(2-\lambda-\rho)^2\psi_{00}+\sum_{r=1}^{\infty}\left(-\rho^{r-1}+2\rho^r-\rho^{r+1}\right)^2\psi_{0r}+\sum_{r=1}^{\infty}\left(-\lambda^{r-1}+2\lambda^r-\lambda^{r+1}\right)^2\psi_{r0}\right]}{4(1-\rho\lambda)^2\sigma_\varepsilon^4})$.
   If ε_{it} and e_{it} are independent, $\sqrt{nT}\left(\widehat{\beta}_{FD}-\beta\right)\Rightarrow N\left(0,\frac{(1+\lambda)^2\left[(2-\rho-\lambda)^2+\frac{(1-\rho)^3}{1+\rho}+\frac{(1-\lambda)^3}{1+\lambda}\right]\sigma_e^2}{4(1-\rho\lambda)^2\sigma_\varepsilon^2}\right)$.

2. If $\rho=1$ and $|\lambda|<1$,
   $\sqrt{nT}\left(\widehat{\beta}_{FD}-\beta\right)\Rightarrow N\left(0,\frac{(1+\lambda)\psi_{00}}{2\sigma_\varepsilon^4}\right)$.
   If ε_{it} and e_{it} are independent, $\sqrt{nT}\left(\widehat{\beta}_{FE}-\beta\right)\Rightarrow N\left(0,\frac{(1+\lambda)\sigma_e^2}{2\sigma_\varepsilon^2}\right)$.

3. If $|\rho| < 1$ and $\lambda = 1$,

   $\sqrt{nT}\left(\widehat{\beta}_{FD} - \beta\right) \Rightarrow N\left(0, \frac{2\psi_{00}}{(1+\rho)\sigma_\varepsilon^4}\right)$.

   If ε_{it} and e_{it} are independent, $\sqrt{nT}\left(\widehat{\beta}_{FD} - \beta\right) \Rightarrow N\left(0, \frac{2\sigma_e^2}{(1+\rho)\sigma_\varepsilon^2}\right)$.

4. If $\rho = 1$ and $\lambda = 1$,

   $\sqrt{nT}\left(\widehat{\beta}_{FD} - \beta\right) \Rightarrow N\left(0, \frac{\sigma_e^2}{\sigma_\varepsilon^2}\right)$.

Corollary 3 follows directly from Theorem 3.

6 GLS Estimator

Let us rewrite equation (1) in vector form

$$\mathbf{y} = \alpha\boldsymbol{\iota}_{nT} + \mathbf{x}\beta + \mathbf{u}$$

where $\mathbf{y}$ is $nT \times 1$, $\mathbf{x}$ is a vector of x_{it} of dimension $nT \times 1$, $\boldsymbol{\iota}_{nT}$ is a vector of ones of dimension nT. and $\mathbf{u}$ is $nT \times 1$. As shown in the Appendix,

$$\widehat{\beta}_{GLS} = \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}\left[\mathbf{x}'\Phi^{-1}\mathbf{y} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y}\right] \tag{13}$$

and

$$\widehat{\beta}_{GLS} - \beta = \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}\left[\mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}\right],$$

where $\Phi = E(\mathbf{u}\mathbf{u}')$.

One can decompose the variance-covariance matrix into

$$\Phi = E(\mathbf{u}\mathbf{u}') = \sigma_\mu^2\left(I_n \otimes \boldsymbol{\iota}_T\boldsymbol{\iota}_T'\right) + \varpi_e^2\left(I_n \otimes \mathbf{A}\right)$$

where $\boldsymbol{\iota}_T$ is a vector of ones of dimension T. $\mathbf{A}$ is the variance-covariance matrix of ν_{it},

$$\mathbf{A} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}$$

when $|\rho| < 1$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & T \end{bmatrix}$$

when $\rho = 1$. Thus, it can be shown that

$$\Phi^{-1} = I_n \otimes \left[\frac{1}{\varpi_e^2} \left(\mathbf{A}^{-1} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{A}^{-1} \boldsymbol{\iota}_T \boldsymbol{\iota}_T' \mathbf{A}^{-1} \right) \right]$$

where $\theta = \boldsymbol{\iota}_T' \mathbf{A}^{-1} \boldsymbol{\iota}_T$.

When $|\rho| < 1$, this estimation is equivalent to the Prais-Winsten transformation method suggested by Baltagi and Li (1991). One can easily verify that $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$, where

$$\mathbf{C} = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -\rho & 1 & 0 \\ 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix}$$

is the Prais-Winsten transformation matrix as in Baltagi and Li (1991).

Thus, we have the following theorem:

Theorem 4 *Under Assumptions* $1 - 2$*, we obtain the following results:*

1. If $|\rho| < 1$ and $|\lambda| < 1$,

   (a) $\left(\widehat{\beta}_{GLS} - \beta \right) \xrightarrow{p} \frac{\lim \frac{1}{T} \sum_{t=1}^{T} E[(x_{it} - \rho x_{it-1}) e_{it}]}{\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2(\lambda - 2\rho\lambda^2 + \rho^2\lambda - \rho)\gamma_\varepsilon^2}{1-\lambda}}$

   (b) $\sqrt{nT} \left(\widehat{\beta}_{GLS} - \beta - \tau_{1nT}^{GLS} \right) \Rightarrow N\left(0, \kappa_1^{GLS}\right)$,

   where $\tau_{1nT}^{GLS} = \frac{\lim \frac{1}{T} \sum_{t=1}^{T} E[(x_{it} - \rho x_{it-1}) e_{it}]}{\sigma_e^2 \frac{1}{nT} X' \Phi^{-1} X}$, $\kappa_1^{GLS} = \frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{(1-\lambda^2)\left[\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho)\gamma_\varepsilon^2}{1-\lambda}\right]^2}$

2. If $\rho = 1$ and $|\lambda| < 1$,

(a) $\left(\widehat{\beta}_{GLS}-\beta\right) \xrightarrow{p} \frac{\lim \frac{1}{n}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T}E[(x_{it}-x_{it-1})e_{it}]}{\frac{2\sigma_{\varepsilon}^{2}}{1+\lambda}+\frac{2(-2\lambda^{2}+2\lambda-1)\gamma_{\varepsilon}^{2}}{1-\lambda}}$,

(b) $\sqrt{nT}\left(\widehat{\beta}_{GLS}-\beta-\tau_{2nT}^{GLS}\right) \Rightarrow N\left(0,\kappa_{2}^{GLS}\right)$,

where $\tau_{2nT}^{GLS}=\frac{\lim \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E[(x_{it}-x_{it-1})e_{it}]}{\sigma_{e}^{2}\frac{1}{nT}X'\Phi^{-1}X}$, $\kappa_{2}^{GLS}=\frac{2\psi_{00}}{(1+\lambda)\left[\frac{2\sigma_{\varepsilon}^{2}}{1+\lambda}+\frac{2(-2\lambda^{2}+2\lambda-1)\gamma_{\varepsilon}^{2}}{1-\lambda}\right]^{2}}$

3. If $|\rho|<1$ and $\lambda=1$,

(a) $T\left(\widehat{\beta}_{GLS}-\beta\right) \xrightarrow{p} \frac{-3\varpi_{\varepsilon e}+6\gamma_{\varepsilon e}+\frac{6}{1-\rho}\sigma_{\varepsilon e}}{(1-\rho)\varpi_{\varepsilon}^{2}}$,

(b) $\sqrt{nT}\left(\widehat{\beta}_{GLS}-\beta\right)-\sqrt{n}\tau_{3nT}^{GLS} \Rightarrow N\left(0,\kappa_{3}^{GLS}\right)$,

where $\tau_{3nT}^{GLS}=\frac{\frac{1}{n}\sum_{i=1}^{n}(1-\rho)\left(\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\bar{x}_{i})\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^{2}}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right)}{\sigma_{e}^{2}\frac{1}{nT}X'\Phi^{-1}X}$, $\kappa_{3}^{GLS}=\frac{6\varpi_{e,\varepsilon}}{(1-\rho)^{2}\varpi_{\varepsilon}^{2}}$.

4. If $\rho=1$ and $\lambda=1$,

(a) $\sqrt{T}\left(\widehat{\beta}_{GLS}-\beta\right) \xrightarrow{p} \frac{\sigma_{\varepsilon e}}{\sigma_{\varepsilon}^{2}}$,

(b) $\sqrt{nT}\left(\widehat{\beta}_{GLS}-\beta\right)-\sqrt{n}\tau_{4nT}^{GLS} \Rightarrow N\left(0,\kappa_{4}^{GLS}\right)$,

where $\tau_{4nT}^{GLS}=\frac{\sqrt{n}\sigma_{\varepsilon e}}{\sigma_{e}^{2}\frac{1}{nT}X'\Phi^{-1}X}$, $\kappa_{4}^{GLS}=\frac{\varpi_{e}^{2}\varpi_{\varepsilon}^{2}}{\sigma_{\varepsilon}^{4}}$.

It is well known that the random effects model imposes the critical assumption that μ_i needs to be independent of x_{it}. It is worth pointing out that this assumption is only needed for the case $\rho<1$. When $\rho=1$, the GLS transformation is identical to the first-difference estimation except for the first observation of each individual. The Cochrane-Orcutt (CO) procedure of GLS transformation ignores the information contained in the first observation. Hence, if one use Cochrane-Orcutt (CO) procedure, the GLS estimation will be the same as the FD estimation and μ_i need not to be independent of x_{it} when $\rho=1$. When $\rho<1$, $E(\mu_i|x_{it})=0$ is required, otherwise $\hat{\beta}_{GLS}$ would be biased and inconsistent. In this case, one may use the within or first-difference transformation to wipe out μ_i and then run GLS estimation. The Within-GLS or FD-GLS estimators can be shown asymptotically as efficient as GLS estimator. However, this is beyond the scope of this paper and can be left as a further extension. The following corollary follows directly from Theorem 4.

Corollary 4 *When $E\left(e_{it}\varepsilon_{i(t+k)}\right)=0$ for all i and k, under the same conditions as for Theorem 4, then*

1. If $|\rho|<1$ and $|\lambda|<1$,

$\sqrt{nT}\left(\widehat{\beta}_{GLS}-\beta\right) \Rightarrow N\left(0,\frac{(1-\lambda^{2})\psi_{00}}{(1-2\rho\lambda+\rho^{2})\sigma_{\varepsilon}^{4}}\right)$.

If ε_{it} and e_{it} are independent, $\sqrt{nT}\left(\widehat{\beta}_{GLS}-\beta\right) \Rightarrow N\left(0,\frac{(1-\lambda^{2})\sigma_{e}^{2}}{(1-2\rho\lambda+\rho^{2})\sigma_{\varepsilon}^{2}}\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,

   $\sqrt{nT}\left(\widehat{\beta}_{GLS} - \beta\right) \Rightarrow N\left(0, \frac{(1+\lambda)\psi_{00}}{2\sigma_\varepsilon^4}\right)$.

   If ε_{it} and e_{it} are independent, $\sqrt{nT}\left(\widehat{\beta}_{GLS} - \beta\right) \Rightarrow N\left(0, \frac{(1+\lambda)\sigma_e^2}{2\sigma_\varepsilon^2}\right)$.

3. If $|\rho| < 1$ and $\lambda = 1$,

   $\sqrt{n}T\left(\widehat{\beta}_{GLS} - \beta\right) \Rightarrow N\left(0, \frac{6\sigma_e^2}{(1-\rho)^2\sigma_\varepsilon^2}\right)$.

4. If $\rho = 1$ and $\lambda = 1$,

   $\sqrt{nT}\left(\widehat{\beta}_{GLS} - \beta\right) \Rightarrow N\left(0, \frac{\sigma_e^2}{\sigma_\varepsilon^2}\right)$.

Case 1 is the textbook result. Case 3 is discussed in Choi (1999). Cases 2 and 4 are new.

7 Feasible GLS Estimator

It is clear that the GLS estimator in Section 6 is not feasible. In this section, we discuss feasible GLS estimation. Assuming $E\left(e_{it}\varepsilon_{i(t+k)}\right) = 0$ for all i and k and ε_{it} and e_{it} are independent, a feasible GLS estimator can be calculated by estimating the autocorrelation coefficient ρ and the variance components σ_μ^2 and σ_e^2. To estimate these parameters, we take the following steps.

First, retrieve the residual estimator $\widehat{\nu}_{it}$ from the FE regression in (1). Now ρ can be estimated as the correlation between $\widehat{\nu}_{it}$ and $\widehat{\nu}_{it-1}$, i.e.,

$$\widehat{\rho} = \frac{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it} - \overline{\widehat{\nu}}\right)\left(\widehat{\nu}_{it-1} - \overline{\widehat{\nu}}\right)}{\sqrt{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it-1} - \overline{\widehat{\nu}}\right)^2}\sqrt{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it-1} - \overline{\widehat{\nu}}\right)^2}}, \tag{14}$$

where $\overline{\widehat{\nu}}$ is the sample average of $\widehat{\nu}_{it}$. Alternatively, as suggested by Baltagi and Li (1991), one can estimate ρ by

$$\widehat{\rho} = \sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it}\widehat{\nu}_{it-1} / \sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it-1}\right)^2.$$

Baltagi and Li (1997) suggests another consistent estimator $\widehat{\rho} = \left(\tilde{Q}_1 - \tilde{Q}_2\right) / \left(\tilde{Q}_0 - \tilde{Q}_1\right)$, where $\tilde{Q}_s = \sum_{i=1}^{n}\sum_{t=s+1}^{T}\widehat{u}_{it}\widehat{u}_{it-s}/n\left(T-s\right)$. We choose the correlation coefficient estimator because it ensures that $\widehat{\rho}$ is always between 0 and 1. It can be shown that $\widehat{\rho}$ in (14) is a consistent estimator of ρ by using the Theorem 2, i.e., $\widehat{\rho} \xrightarrow{p} \rho$ if $|\rho| < 1$.

Next, using the FE residuals $\widehat{\nu}_{it}$ and the estimate of the autocorrelation coefficient $\widehat{\rho}$, we can get $\widehat{e}_{it}$. Therefore σ_e^2 can be estimated by $\hat{\sigma}_e^2 = \frac{1}{nT}\sum_{i=1}^{N}\sum_{t=1}^{T}\widehat{e}_{it}^2$. Also, σ_μ^2 can be estimated using $\hat{\sigma}_\mu^2 =$

$\frac{1}{nT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{u}_{it}^2-\widehat{\nu}_{it}^2\right)$, where $\widehat{u}_{it}$ denote the OLS residuals from equation (10). $\hat{\sigma}_e^2$ and $\hat{\sigma}_\mu^2$ are consistent estimators for σ_e^2 and σ_μ^2 respectively, i.e., $\hat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2$, $\hat{\sigma}_\mu^2 \xrightarrow{p} \sigma_\mu^2$ if $|\rho|<1$.These variance components can be estimated by using the variance decomposition and the Prais-Winsten (PW) transformation suggested by Baltagi and Li (1991). Alternatively, one can also use the Cochrane-Orcutt (CO) procedure, which ignores the information contained in the first observation. As suggested by Maeshiro (1976), Beach and MacKinnon (1978) and Park and Mitchell (1980), estimation using the PW transformation is more efficient than using the CO procedure when the regressors are trended.

When the assumptions of corollary 4 hold, one can show that feasible GLS has the same asymptotic distribution as true GLS. Define $\phi=\left(\rho,\sigma_\mu^2,\sigma_e^2\right)$ and $\hat{\phi}$ is its corresponding estimator. Then $\Phi=\Phi(\phi)$. Further define that $G_k(\phi)=\partial\Phi^{-1}(\phi)/\partial\phi_k$,where $k=1,2,3$. For example, in case 1, a Taylor's series expansion as in Fuller and Battese (1973) gives

$$
\begin{aligned}
&\sqrt{nT}\left(\hat{\tau}_{GLS}-\tau\right)\\
=&\left(\frac{Z'\Phi^{-1}\left(\hat{\phi}\right)Z}{nT}\right)^{-1}\left(\frac{Z'\Phi^{-1}\left(\hat{\phi}\right)u}{\sqrt{nT}}\right)\\
=&\left(\frac{Z'\Phi^{-1}(\phi)Z}{nT}\right)^{-1}\left(\frac{Z'\Phi^{-1}(\phi)u}{\sqrt{nT}}\right)+\sum_{k=1}^{3}[\left(\frac{Z'\Phi^{-1}(\phi^*)Z}{nT}\right)^{-1}\left(\frac{Z'G_k(\phi^*)u}{\sqrt{nT}}\right)\\
&-\left(\frac{Z'\Phi^{-1}(\phi^*)Z}{nT}\right)^{-1}\left(\frac{Z'G_k(\phi^*)Z}{nT}\right)\left(\frac{Z'\Phi^{-1}(\phi^*)Z}{nT}\right)^{-1}\left(\frac{Z'\Phi^{-1}(\phi^*)u}{\sqrt{nT}}\right)]\left(\hat{\phi}-\phi\right)\\
=&\left(\frac{Z'\Phi^{-1}(\phi)Z}{nT}\right)^{-1}\left(\frac{Z'\Phi^{-1}(\phi)u}{\sqrt{nT}}\right)+o_p(1)
\end{aligned}
$$

where ϕ^* lies between $\hat{\phi}$ and ϕ, hence $\phi^*\xrightarrow{p}\phi$. The last equal sign holds if $\frac{Z'\Phi^{-1}(\phi)Z}{nT}=O_p(1)$, $\frac{Z'G_k(\phi^*)Z}{nT}=O_p(1)$, $\frac{Z'\Phi^{-1}(\phi^*)u}{\sqrt{nT}}=O_p(1)$ and $\frac{Z'G_k(\phi^*)u}{\sqrt{nT}}=O_p(1)$. This follows using similar arguments in the proofs of the Theorems above. The proofs are similar for the remaining three cases and are omitted to save space.

8 Efficiency Comparisons

This section summarizes the relative efficiency of OLS, FE, GLS and FD estimators. First, the speed of convergence for the different cases considered are summarized as follows:

	OLS	FE	FD	GLS
Case 1: $\|\rho\| < 1$ and $\|\lambda\| < 1$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$
Case 2: $\rho = 1$ and $\|\lambda\| < 1$	$\sqrt{n}$	$\sqrt{n}$	$\sqrt{nT}$	$\sqrt{nT}$
Case 3: $\|\rho\| < 1$ and $\lambda = 1$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$
Case 4: $\rho = 1$ and $\lambda = 1$	$\sqrt{n}$	$\sqrt{n}$	$\sqrt{nT}$	$\sqrt{nT}$

In case 1, the four estimators have the same convergence speed of $\sqrt{nT}$. The efficiency of the OLS estimator is hard to compare with the remaining estimators because OLS does not difference out μ_i, and as a result its variance still contains σ_μ^2. That GLS is more efficient than FE and FD is evident from the Gauss-Markov theorem. Since these estimators all converge at same rate $\sqrt{nT}$, we plot the relative efficiency of the FE and FD estimators with respect to true GLS in Figure 1 and 2. The relative efficiency of the FE estimator with respect to true GLS is given by

$$
\begin{aligned}
var\left(\widehat{\beta}_{FE}\right)/var\left(\widehat{\beta}_{GLS}\right) &= \frac{(1+\rho\lambda)\left(1-\lambda^2\right)\sigma_e^2}{(1-\rho\lambda)\left(1-\rho^2\right)\sigma_\varepsilon^2} \Big/ \frac{\left(1-\lambda^2\right)\sigma_e^2}{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2} \\
&= \frac{(1+\rho\lambda)\left(1-2\rho\lambda+\rho^2\right)}{(1-\rho\lambda)\left(1-\rho^2\right)}.
\end{aligned}
$$

The relative efficiency of the FD estimator with respect to true GLS is given by

$$
\begin{aligned}
var\left(\widehat{\beta}_{FD}\right)/var\left(\widehat{\beta}_{GLS}\right) &= \frac{(1+\lambda)^2\left[(2-\rho-\lambda)^2+\frac{(1-\rho)^3}{1+\rho}+\frac{(1-\lambda)^3}{1+\lambda}\right]\sigma_e^2}{4(1-\rho\lambda)^2\sigma_\varepsilon^2} \Big/ \frac{\left(1-\lambda^2\right)\sigma_e^2}{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2} \\
&= \frac{(1+\lambda)\left(1-2\rho\lambda+\rho^2\right)\left[(2-\rho-\lambda)^2+\frac{(1-\rho)^3}{1+\rho}+\frac{(1-\lambda)^3}{1+\lambda}\right]}{4(1-\lambda)(1-\rho\lambda)^2}.
\end{aligned}
$$

One can easily verify that both relative efficiencies are larger or equal to 1. Comparing the GLS estimator with the FE and FD estimators, the relative efficiency depends on the values of ρ and λ. As shown in Figure 1 and 2, when ρ is small, the FE estimator performs well in terms of relative efficiency with respect to true GLS. When ρ is large, the FD estimators performs well in terms of relative efficiency with respect to true GLS.

In case 2, the disturbance is $I(1)$ but the regressor is I(0). The noise is strong so that it dominates the signal. In the time series case, the OLS estimator is not consistent. After double smoothing using panel data, the asymptotic distribution becomes normal and the convergence speed is $\sqrt{n}$. GLS estimation, however, transforms the disturbance into $I(0)$. Therefore the convergence speed is $\sqrt{nT}$. When the disturbance is $I(1)$, first-difference estimation will be the same as GLS except for the first observation. Hence it is also $\sqrt{nT}$ consistent.

In case 3, the disturbance is $I(0)$ but the regressor is $I(1)$. This is the cointegration case. The cointegration literature shows that the GLS estimators is T consistent in time series models. In the panel data model, both GLS and FE are $\sqrt{n}T$ consistent.

In case 4, both the disturbance and the regressor are $I(1)$. This is the spurious regression case. As shown in Kao (1999), the FE estimator is $\sqrt{n}$ consistent. For the same reason given in case 2, first-differencing transforms the disturbance term from $I(1)$ to $I(0)$. Therefore, the convergence speed of both the GLS or the FD estimators is $\sqrt{nT}$.

In case 3, the FE estimator is more efficient than the FD estimator when v_{it} are stationary, including the special case when v_{it} are serially uncorrelated. In cases 2 and 4, the FD estimator is more efficient when v_{it} follows a random walk. These results verify the conclusion in Wooldridge (2002). However, in case 1, when ρ is large, even though v_{it} does not follow a random walk, the FD estimator is still more efficient than the FE estimator.

9 Monte Carlo Simulation

This section reports the results of Monte Carlo experiments designed to investigate the finite sample relative efficiency of the OLS, FE, FD, GLS-CO, GLS-PW estimators with respect to true GLS. The model is generated by

$$y_{it} = x_{it}\beta + \mu_i + v_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \tag{15}$$

with $\beta = 10$, $\mu_i \overset{iid}{\sim} N(0, 5)$ and v_{it} and x_{it} follow an AR(1) process given in (2) and (3), respectively with ρ and λ varying over the range $(0, 0.2, 0.4, 0.6, 0.8, 0.9, 1)$ and $\sigma_\varepsilon^2 = \sigma_e^2 = 5$. The sample sizes n and T are varied over the range $(20, 40, 60, 120, 240)$. For each experiment, we perform $10,000$ replications. For each replication we estimate the model using OLS, FE, FD, GLS-CO, GLS-PW and true GLS. Even with this modest design we had 1225 experiments. GAUSS for Windows 6.0 was used to perform the simulations. Random numbers for μ_i and ε_{it} were generated by the GAUSS procedure RNDNS. We generated $n(T+1000)$ random numbers and then split them into n series so that each series had the same mean and variance. The first $1,000$ observations were discarded for each series.

Tables 1-3 give the relative mean square error (MSE) of each estimator of β with respect to true GLS for various values of ρ, λ, n, and T. We only report 3 Tables to give a flavour of the results, the rest are available upon request from the authors. Several conclusions emerge from these results. First, the true GLS estimator is the most efficient one in terms of mean squared error. Its efficiency gain over the OLS estimator is enormous particularly when ρ and/or λ is large. Second, the FE estimator is less efficient than true GLS, but more efficient than the feasible GLS estimator when $\rho = 0$. However, when ρ increases, the feasible GLS estimator quickly becomes more efficient than the FE estimator. Third, the FD estimator is also less efficient than true GLS. When ρ increases, the FD estimator becomes as efficient as the GLS estimator. Interestingly, the FD estimator behaves poorly when λ is close to 1 but ρ is small. Fourth, the feasible GLS estimator is slightly less efficient than the true GLS estimator and beats OLS, FE and FD as long as $\rho > 0.2$. In summary, our simulation results show that the feasible GLS estimator performs well, and is second best only to true GLS when $\rho > 0.2$

10 Conclusion

In this paper, we compared the efficiency of OLS, FE, FD, and GLS estimators in panel models with $I(0)$ and $I(1)$ regressor and regression error. When the regression error is $I(0)$ and the regressor is $I(1)$ and hence the model is cointegrated, both the FE and GLS estimators are asymptotically efficient. When the regression error is $I(1)$ and the regressor is $I(1)$ and hence the model is spurious, the FE and GLS estimators are $\sqrt{n}$ and$\sqrt{nT}$ consistent, respectively. This implies that GLS is the preferred estimator as far as the regression error specification is concerned since GLS converges at as good or better rate in both cases (i.e., regression error is either $I(0)$ or $I(1)$).

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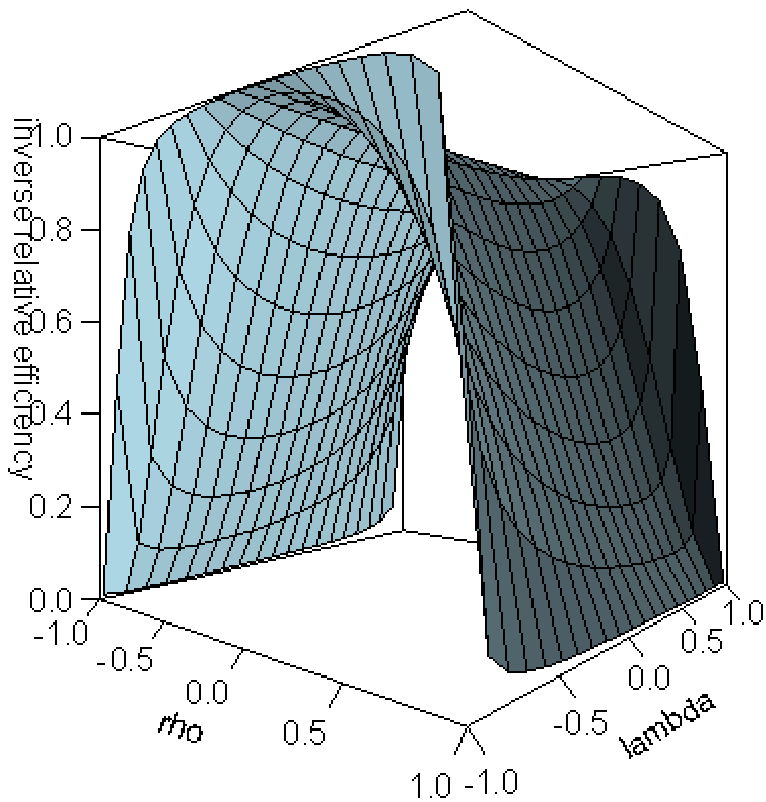

Figure 1: Relative Efficiency of GLS to FE Estimator

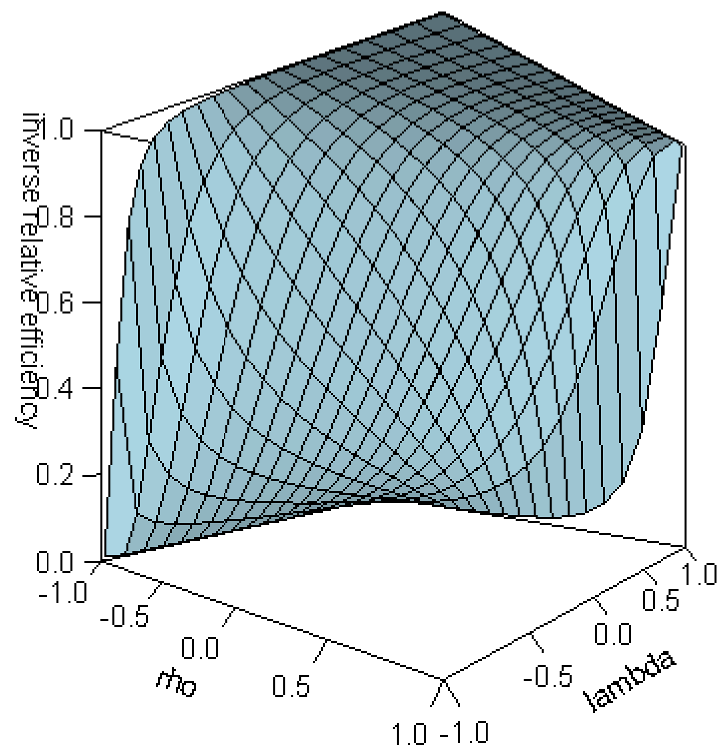

Figure 2: Relative Efficiency of GLS to FD Estimator

Table 1: Relative Efficiencies of Standard Panel Data Estimators (N = 40, T= 20)

	ρ	λ						
		0	0.2	0.4	0.6	0.8	0.9	1
OLS	0	1.915	2.310	2.897	3.838	5.305	5.799	5.211
	0.2	2.077	2.404	2.845	3.496	4.436	4.668	4.065
	0.4	2.512	2.815	3.154	3.565	4.032	3.994	3.273
	0.6	3.427	3.793	4.108	4.340	4.346	3.956	2.906
	0.8	5.784	6.476	7.017	7.260	6.745	5.653	3.506
	0.9	9.257	10.529	11.590	12.198	11.475	9.528	5.451
	1	22.685	26.255	29.502	31.992	31.608	27.087	15.126
FE	0	1.002	1.003	1.005	1.009	1.024	1.048	1.102
	0.2	1.073	1.075	1.073	1.071	1.076	1.098	1.161
	0.4	1.315	1.332	1.332	1.311	1.278	1.281	1.330
	0.6	1.838	1.915	1.949	1.918	1.811	1.755	1.733
	0.8	2.970	3.223	3.397	3.426	3.198	2.972	2.650
	0.9	4.066	4.505	4.843	4.976	4.678	4.294	3.619
	1	7.306	8.326	9.222	9.843	9.767	9.330	8.142
FD	0	1.486	1.699	2.021	2.561	3.601	4.594	7.548
	0.2	1.223	1.329	1.494	1.780	2.355	2.934	4.722
	0.4	1.095	1.141	1.215	1.350	1.642	1.964	3.022
	0.6	1.037	1.053	1.079	1.129	1.253	1.413	1.991
	0.8	1.013	1.016	1.022	1.032	1.064	1.117	1.342
	0.9	1.006	1.007	1.009	1.011	1.020	1.036	1.117
	1	1.016	1.019	1.021	1.022	1.023	1.023	1.024
GLS-PW	0	1.272	1.357	1.456	1.565	1.648	1.652	1.625
	0.2	1.120	1.168	1.232	1.316	1.410	1.440	1.453
	0.4	1.043	1.063	1.092	1.137	1.201	1.234	1.270
	0.6	1.012	1.017	1.025	1.039	1.065	1.085	1.119
	0.8	1.002	1.002	1.003	1.004	1.008	1.014	1.027
	0.9	1.000	1.000	1.000	1.000	1.002	1.006	1.021
	1	1.016	1.018	1.020	1.024	1.031	1.045	1.111

Notes: (a) Relative mean square error with respect to the true GLS. (b) 10,000 replications. $\sigma_\mu^2 = \sigma_e^2 = 5$.

Table 2: Relative Efficiencies of Standard Panel Data Estimators (N = 60, T= 60)

	ρ	λ						
		0	0.2	0.4	0.6	0.8	0.9	1
OLS	0	1.963	2.414	3.123	4.441	7.826	12.201	13.958
	0.2	2.101	2.467	2.989	3.881	6.062	8.836	9.628
	0.4	2.523	2.858	3.249	3.796	4.980	6.446	6.329
	0.6	3.445	3.848	4.207	4.509	4.844	5.209	4.077
	0.8	6.008	6.802	7.438	7.754	7.290	6.375	3.150
	0.9	10.598	12.223	13.622	14.515	13.850	11.702	4.367
	1	61.715	73.063	84.083	94.300	100.976	97.379	41.327
FE	0	1.000	1.000	1.000	1.000	1.002	1.005	1.037
	0.2	1.081	1.083	1.078	1.064	1.039	1.028	1.062
	0.4	1.359	1.379	1.371	1.323	1.219	1.151	1.149
	0.6	2.010	2.110	2.147	2.073	1.802	1.576	1.415
	0.8	3.871	4.291	4.601	4.681	4.180	3.485	2.511
	0.9	6.686	7.647	8.490	9.050	8.675	7.464	4.664
	1	20.601	24.289	27.902	31.303	33.476	32.129	23.459
FD	0	1.485	1.714	2.079	2.779	4.672	7.440	21.119
	0.2	1.211	1.322	1.505	1.864	2.861	4.347	11.967
	0.4	1.082	1.128	1.206	1.367	1.834	2.559	6.559
	0.6	1.026	1.041	1.067	1.121	1.292	1.579	3.449
	0.8	1.007	1.009	1.014	1.023	1.056	1.123	1.768
	0.9	1.002	1.003	1.004	1.007	1.014	1.030	1.267
	1	1.006	1.007	1.008	1.009	1.010	1.010	1.010
GLS-PW	0	1.293	1.392	1.512	1.660	1.820	1.834	1.667
	0.2	1.126	1.181	1.258	1.376	1.564	1.663	1.612
	0.4	1.043	1.066	1.101	1.162	1.291	1.399	1.455
	0.6	1.011	1.017	1.026	1.045	1.095	1.155	1.246
	0.8	1.001	1.002	1.003	1.005	1.010	1.021	1.062
	0.9	1.000	1.000	1.000	1.000	1.001	1.002	1.012
	1	1.005	1.006	1.007	1.008	1.009	1.010	1.046

Notes: (a) Relative mean square error with respect to the true GLS. (b) 10,000 replications. $\sigma_\mu^2 = \sigma_e^2 = 5$.

Table 3: Relative Efficiencies of Standard Panel Data Estimators (N = 240, T= 60)

	ρ	λ						
		0	0.2	0.4	0.6	0.8	0.9	1
OLS	0	1.903	2.330	3.022	4.305	7.364	11.040	12.103
	0.2	2.042	2.390	2.895	3.771	5.770	8.095	8.331
	0.4	2.462	2.783	3.152	3.690	4.802	6.021	5.490
	0.6	3.385	3.783	4.110	4.392	4.731	5.003	3.598
	0.8	5.877	6.679	7.269	7.545	7.169	6.308	2.932
	0.9	10.283	11.912	13.242	14.094	13.687	11.809	4.236
	1	66.063	78.409	89.876	100.477	108.863	107.078	43.764
FE	0	1.000	1.001	1.001	1.002	1.006	1.011	1.024
	0.2	1.085	1.088	1.082	1.073	1.065	1.060	1.054
	0.4	1.369	1.395	1.384	1.344	1.274	1.219	1.147
	0.6	2.036	2.152	2.182	2.113	1.912	1.718	1.433
	0.8	3.850	4.301	4.598	4.683	4.352	3.789	2.601
	0.9	6.426	7.390	8.173	8.699	8.595	7.716	4.808
	1	19.538	23.082	26.375	29.349	31.405	30.910	26.383
FD	0	1.471	1.687	2.051	2.733	4.389	6.646	18.377
	0.2	1.210	1.312	1.495	1.853	2.748	3.973	10.465
	0.4	1.085	1.125	1.203	1.368	1.803	2.409	5.791
	0.6	1.030	1.041	1.065	1.123	1.289	1.536	3.103
	0.8	1.009	1.011	1.014	1.024	1.059	1.120	1.675
	0.9	1.004	1.004	1.005	1.007	1.015	1.030	1.241
	1	0.999	0.998	0.997	0.996	0.998	0.998	1.000
GLS-PW	0	1.290	1.384	1.508	1.647	1.714	1.640	1.527
	0.2	1.128	1.179	1.258	1.374	1.504	1.523	1.473
	0.4	1.047	1.066	1.102	1.165	1.268	1.321	1.336
	0.6	1.014	1.018	1.027	1.047	1.091	1.125	1.163
	0.8	1.003	1.003	1.004	1.006	1.013	1.019	1.033
	0.9	1.001	1.001	1.001	1.001	1.003	1.004	1.008
	1	0.999	0.998	0.997	0.997	0.999	1.002	1.045

Notes: (a) Relative mean square error with respect to the true GLS. (b) 10,000 replications. $\sigma_\mu^2 = \sigma_e^2 = 5$.

Appendix

A Proof of Theorem 1

The following lemmas are needed to prove Theorem 1. All limits are taken as $T \to \infty$ and followed by $n \to \infty$ sequentially, $(n, T) \overset{\text{seq}}{\to} \infty$.

Lemma 1 *If Assumptions* $1-2$ *hold, then*

1. If $|\lambda| < 1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \overset{p}{\to} \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$.
2. If $\lambda = 1$, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \overset{p}{\to} \frac{\varpi_\varepsilon^2}{2}$.

Proof. Consider (1). For a fixed n, it is clear to see that

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}^2\right]-\left(\overline{\overline{x}}\right)^2 \\
\overset{p}{\to} & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}\right]-\frac{1}{n}\sum_{i=1}^{n}E\left(x_{it}\right) \\
= & \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}
\end{aligned}$$

as $T \to \infty$ because

$$\begin{aligned}
E\left(x_{it}^2\right) &= E\left(\sum_{j=0}^{\infty}\lambda^j\varepsilon_{i(t-j)}\right)^2 \\
&= \sum_{j=0}^{\infty}E\left(\lambda^j\varepsilon_{i(t-j)}\right)^2+\sum_{j=0}^{\infty}\sum_{k=0,k\neq j}^{\infty}E\left(\lambda^{j+k}\varepsilon_{i(t-j)}\varepsilon_{i(t-k)}\right) \\
&= \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}
\end{aligned}$$

and $\overline{\overline{x}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{it} \overset{p}{\to} \frac{1}{n}\sum_{i=1}^{n}E\left(x_{it}\right)=0$ as $T \to \infty$. Note $E\left(x_{it}\right)=0$ for all i and t since there is no non-zero drift in (**??**). Then obviously,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \overset{p}{\to} \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$$

holds for all n and hence it holds for a large n as well. This proves (1).

Next we consider (2). Similarly for a fixed n

$$\begin{aligned}
& \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}x_{it}^2\right]-\frac{1}{T}\overline{\overline{x}}^2 \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{x_{it}}{\sqrt{T}}\right)^2\right]-\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{x_{it}}{\sqrt{T}}\right)\right]^2 \\
\Rightarrow & \frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int W_i^2\right)-\left[\frac{1}{n}\sum_{i=1}^{n}\varpi_\varepsilon\int W_i\right]^2
\end{aligned}$$

as $T\to\infty$ because $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{x_{it}}{\sqrt{T}}\right)^2\Rightarrow\varpi_\varepsilon^2\int W_i^2$ and $\frac{1}{T}\sum_{t=1}^{T}\frac{x_{it}}{\sqrt{T}}\Rightarrow\varpi_\varepsilon\int W_i$. Then

$$\frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int W_i^2\right)-\left[\frac{1}{n}\sum_{i=1}^{n}\varpi_\varepsilon\int W_i\right]^2\xrightarrow{p}\frac{\varpi_\varepsilon^2}{2}.$$

as $n\to\infty$ by a law of large numbers (LLN). This is because

$$E\int W_i^2=\frac{1}{2}$$

and

$$E\int W_i=0.$$

This proves (2). ■

Lemma 2 *If Assumptions* $1-3$ *hold, then*

1. If $|\rho|<1$ and $|\lambda|<1$,

   (a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)$

   (b)

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
\Rightarrow & N\left(0,\frac{\sigma_\mu^2\varpi_\varepsilon^2}{\left(1-\lambda\right)^2}+\frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{\left(1-\rho\lambda\right)^2}\right)
\end{aligned}$$

where $\psi_{0r} = E\left(\varepsilon_{i(t-r)}^2 e_{it}^2\right), \psi_{r0} = E\left(\varepsilon_{it}^2 e_{i(t-r)}^2\right), \psi_{00} = E\left(\varepsilon_{it}^2 e_{it}^2\right).$

2. If $\rho = 1$ and $|\lambda| < 1$,

   (a) $\frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right) u_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda}$

   (b) $\frac{1}{\sqrt{n}T}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right) u_{it} - \sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{T}\sum_{t=1}^T \varpi_{e\varepsilon}\nu_{i(t-1)}e_{it}\right)+\delta_{e\varepsilon}}{1-\lambda}\right] \Rightarrow N\left(0, \frac{\varpi_{\varepsilon.e}\varpi_e^2}{2(1-\lambda)^2}\right).$

3. If $|\rho| < 1$ and $\lambda = 1$,

   (a) $\frac{1}{nT^{3/2}}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right) u_{it} \xrightarrow{p} 0$

   (b) $\frac{1}{\sqrt{n}T^{3/2}}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right) u_{it} \Rightarrow N\left(0, \frac{\sigma_\mu^2 \varpi_\varepsilon^2}{3}\right).$

4. If $\rho = 1$ and $\lambda = 1$,

   (a) $\frac{1}{nT^2}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right) u_{it} \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$

   (b) $\frac{1}{\sqrt{n}T^2}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right) u_{it} - \sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{T}\sum_{t=1}^T x_{i(t-1)}\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2} + \delta_{\varepsilon e}\right] \Rightarrow N\left(0, \frac{\varpi_{e.\varepsilon}\varpi_\varepsilon^2}{6}\right).$

Proof. :

Consider (1). For part (a), we note

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right) u_{it} \\
= \; & \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T \left(x_{it} - \overline{\overline{x}}\right)\left(\mu_i + \nu_{it}\right) \\
= \; & \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^n \left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T x_{it}\right)\mu_i\right] - \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{\sqrt{T}}\sum_{t=1}^T x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^n \mu_i\right) \\
& + \frac{1}{n}\sum_{i=1}^n \left[\frac{1}{T}\sum_{t=1}^T x_{it}\nu_{it}\right] - \left(\frac{1}{n}\sum_{i=1}^n \frac{1}{T}\sum_{t=1}^T x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{T}\sum_{t=1}^T \nu_{it}\right) \\
= \; & I - II + III - IV.
\end{aligned}$$

Consider I. It is easy to see that for a fixed n,

$$\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]=\frac{1}{n}\sum_{i=1}^{n}Z_i\mu_i$$

as $T\to\infty$ where

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\Rightarrow Z_i\sim N\left(0,\frac{\varpi_{\varepsilon}^{2}}{(1-\lambda)^{2}}\right)$$

by a central limit theorem (CLT) since $E(x_{it})=0$. Then

$$\frac{1}{n}\sum_{i=1}^{n}Z_i\mu_i=o_p(1)$$

as $n\to\infty$ by a LLN and the assumption that μ_i and x_{it} are uncorrected as in (**??**). Hence $I=\frac{1}{\sqrt{T}}o_p(1)$. Consider II. For a fixed n,

$$\begin{aligned}&\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\\ =&\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\end{aligned}$$

as $T\to\infty$. Then clearly

$$\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\overset{p}{\to}0$$

as $n\to\infty$ because $E(\mu_i)=0$. This proves that

$$II=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)=o_p(1).$$

Next we consider III. Clearly

$$\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]=\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E(x_{it}\nu_{it})+o_p(1)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. Also it is easy to see that

$$IV=o_p(1).$$

Collecting $I-IV$ we then prove (a).

For part (b), for a fixed n,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
= & \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_i+\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]-\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right) \\
& +\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\right] \\
& -\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right) \\
= & I-II+III-IV.
\end{aligned}$$

For I,

$$I=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]\Rightarrow\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z_i\mu_i$$

as $T\to\infty$, and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z_i\mu_i\Rightarrow N\left(0,\frac{\sigma_\mu^2\varpi_\varepsilon^2}{(1-\lambda)^2}\right)$$

as $n\to\infty$ by a CLT. For III,

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
= & \sqrt{nT}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}\nu_{it}-\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\right) \\
\Rightarrow & N\left(0,\frac{\psi_{i00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{i0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{ir0}}{(1-\rho\lambda)^2}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$, which is based on Lemma A0 in Choi(1999). It is easy to see that

$$II=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right)=o_p(1)\,O_p(1)\,.$$

$$IV=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)=\frac{1}{\sqrt{T}}o_p(1)\,O_p(1)=o_p(1)\,.$$

Hence, we have

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)$$
$$\Rightarrow\quad N\left(0,\frac{\sigma_{\mu}^{2}\varpi_{\varepsilon}^{2}}{\left(1-\lambda\right)^{2}}+\frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{\left(1-\rho\lambda\right)^{2}}\right)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. This proves (b).

Consider (2). For part (a), note

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\\
=\;&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_{i}+\nu_{it}\right)\\
=\;&\frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_{i}\right]-\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_{i}\right)\\
&+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]-\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\frac{\nu_{it}}{\sqrt{T}}\right)\\
=\;&I-II+III-IV.
\end{aligned}$$

First we consider III. Note

$$\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\Rightarrow\frac{1}{1-\lambda}\left[\varpi_{e}\varpi_{\varepsilon.e}^{1/2}\left(\int V_{i}dW_{i}\right)+\varpi_{e\varepsilon}\left(\int V_{i}dV_{i}\right)+\delta_{e\varepsilon}\right]$$

as $T\to\infty$. The above is taken from Lemma 1(a) in Kao and Chiang (2000). Then

$$\begin{aligned}
III\;&=\;\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]\\
&=\;\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-\lambda}\left[\varpi_{e}\varpi_{\varepsilon.e}^{1/2}\left(\int V_{i}dW_{i}\right)+\varpi_{e\varepsilon}\left(\int V_{i}dV_{i}\right)+\delta_{e\varepsilon}\right]\\
&\overset{p}{\to}\;\frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda}
\end{aligned}$$

as $n\to\infty$ because $E\left(\int V_{i}dW_{i}\right)=0$ and $E\left(\int V_{i}dV_{i}\right)=-\frac{1}{2}$. It is clear to see that

$$I=\frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_{i}\right]=\frac{1}{\sqrt{T}}o_{p}\left(1\right),$$

$$II=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)=\frac{1}{\sqrt{T}}o_p\left(1\right)$$

and

$$IV=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\frac{\nu_{it}}{\sqrt{T}}\right)=o_p\left(1\right)$$

because $\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\Rightarrow\int V_i$ and $E\left(\int V_i\right)=0$. This proves (a).

For part (b),

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\\
=&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_i+\nu_{it}\right)\\
=&\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]-\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right)\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]-\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right)\\
=&I-II+III-IV.
\end{aligned}$$

First consider III,

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\varpi_{e\varepsilon}\nu_{i,t-1}e_{it}\right)+\delta_{e\varepsilon}}{1-\lambda}\right]\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{1-\lambda}\left[\varpi_e\varpi_{\varepsilon.e}^{1/2}\left(\int V_idW_i\right)+\varpi_{e\varepsilon}\left(\int V_idV_i\right)+\delta_{e\varepsilon}\right]\\
&-\sqrt{n}\left[\frac{\frac{1}{n}\sum_{i=1}^{n}\left(\varpi_{e\varepsilon}\int V_idV_i\right)+\delta_{e\varepsilon}}{1-\lambda}\right]+o_p\left(1\right)\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{1-\lambda}\left[\varpi_e\varpi_{\varepsilon.e}^{1/2}\left(\int V_idW_i\right)\right]+o_p\left(1\right)\\
\Rightarrow&N\left(0,\frac{\varpi_{\varepsilon.e}\varpi_e^2}{2\left(1-\lambda\right)^2}\right),
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$. It is easy to see that

$$I=\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]=\frac{1}{\sqrt{T}}O_p\left(1\right),$$

$$II=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right)=\frac{1}{\sqrt{T}}o_p\left(1\right)O_p\left(1\right)$$

and

$$IV=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right)=o_p\left(1\right)O_p\left(1\right).$$

Hence, we have

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\varpi_{e\varepsilon}\nu_{i(t-1)}e_{it}\right)+\delta_{e\varepsilon}}{1-\lambda}\right]\\ \Rightarrow\ & N\left(0,\frac{\varpi_{\varepsilon.e}\varpi_e^2}{2\left(1-\lambda\right)^2}\right),\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$.

Consider (3). For part (a), it is easy to see that

$$\begin{aligned}&\frac{1}{nT^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\\ =\ &\frac{1}{nT^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_i+\nu_{it}\right)\\ =\ &\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]-\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\\ &+\frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right)\right]-\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\\ =\ &I-II+III-IV.\end{aligned}$$

For a fixed n,

$$\begin{aligned}I\ &=\ \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]\\ &\Rightarrow\ \frac{1}{n}\sum_{i=1}^{n}\left[\left(\varpi_\varepsilon\int W_i\right)\mu_i\right]\end{aligned}$$

as $T\rightarrow\infty$ by a CLT. As $n\rightarrow\infty$ by a LLN and the assumption that μ_i and x_{it} are uncorrected as in (**??**), we have $I=o_p\left(1\right)$. It is easy to show that

$$II=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)=o_p\left(1\right)O_p\left(1\right),$$

$$III = \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{1}{T} \sum_{t=1}^{T} x_{it} \nu_{it} \right) \right] = \frac{1}{\sqrt{T}} o_p(1)$$

and

$$IV = \frac{1}{\sqrt{T}} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \nu_{it} \right) = \frac{1}{\sqrt{T}} o_p(1)\, o_p(1)$$

as $(n, T) \stackrel{\text{seq}}{\rightarrow} \infty$ because $\frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \Rightarrow \varpi_\varepsilon \int W_i$, $E\left[\varpi_\varepsilon \int W_i\right] = 0$, $E[\mu_i] = 0$, $\frac{1}{T} \sum_{t=1}^{T} x_{it} \nu_{it} \Rightarrow \varpi_\varepsilon \varpi_e \int W_i dV_i$, $E\left[\varpi_\varepsilon \varpi_e \int W_i dV_i\right] = 0$, and $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \nu_{it} \Rightarrow N\left(0, \frac{\varpi_e^2}{(1-\rho)^2}\right)$. Hence, we have

$$\frac{1}{nT^{3/2}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}}\right) u_{it} \stackrel{p}{\rightarrow} 0$$

as $(n, T) \stackrel{\text{seq}}{\rightarrow} \infty$.

For part (b), note that

$$\begin{aligned}
& \frac{1}{n^{1/2}T^{3/2}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}}\right) u_{it} \\
= & \frac{1}{n^{1/2}T^{3/2}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}}\right) (\mu_i + \nu_{it}) \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \mu_i \right] - \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mu_i \right) \\
& + \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{T} \sum_{t=1}^{T} x_{it} \nu_{it} \right] - \frac{1}{\sqrt{T}} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \nu_{it} \right) \\
= & I - II + III - IV.
\end{aligned}$$

For a fixed n,

$$\begin{aligned}
I &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \mu_i \right] \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\varpi_\varepsilon \int W_i \right) \mu_i \right]
\end{aligned}$$

as $T \rightarrow \infty$, and

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\varpi_\varepsilon \int W_i \right) \mu_i \right] \Rightarrow N\left(0, \frac{\sigma_\mu^2 \varpi_\varepsilon^2}{3}\right)$$

as $n \to \infty$ by a CLT with $E\left[\varpi_\varepsilon \int W_i\right] = 0$ and $Var\left[\varpi_\varepsilon \int W_i\right] = \frac{1}{3}\varpi_\varepsilon^2$. It is easy to show that

$$II = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right) = o_p(1)\,O_p(1),$$

$$III = \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right] = \frac{1}{\sqrt{T}}O_p(1),$$

and

$$IV = \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right) = \frac{1}{\sqrt{T}}o_p(1)\,O_p(1)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. Hence, we have

$$\frac{1}{n^{1/2}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it} \Rightarrow N\left(0,\frac{\sigma_\mu^2\varpi_\varepsilon^2}{3}\right)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Consider (4). For part (a), it is easy to see that

$$\begin{aligned}
& \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it} \\
= & \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_i+\nu_{it}\right) \\
= & \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right] - \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right) \\
& +\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it}\right] - \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right) \\
= & I - II + III - IV.
\end{aligned}$$

For a fixed n,

$$\begin{aligned}
III &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it}\right] \\
&\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\int W_iV_i + \varpi_{\varepsilon e}\left(\int W_i^2\right) + \delta_{\varepsilon e}\right]
\end{aligned}$$

as $T \to \infty$ by a CLT. We then have

$$\frac{1}{n}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_i V_i + \varpi_{\varepsilon e}\left(\int W_i^2\right) + \delta_{\varepsilon e}\right] \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}.$$

as $n \to \infty$. So

$$III \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. Again

$$I = \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right] = \frac{1}{\sqrt{T}}o_p(1)$$

$$II = \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right) = \frac{1}{\sqrt{T}}o_p(1)\,o_p(1)$$

and

$$IV = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right) = o_p(1)\,o_p(1)$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. Hence, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it} \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

For part (b),

$$\begin{aligned}
&\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it} \\
=\;& \frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_i+\nu_{it}\right) \\
=\;& \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right] - \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right) \\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it} - \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right) \\
=\;& I - II + III - IV.
\end{aligned}$$

For a fixed n,

$$\begin{aligned} III &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it} \\ &\Rightarrow \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_iV_i + \varpi_{\varepsilon e}\left(\int W_i^2\right)+\delta_{\varepsilon e}\right] \end{aligned}$$

as $T \to \infty$ by a CLT. As $(n,T) \overset{\text{seq}}{\to} \infty$, we have

$$\begin{aligned} & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it} - \sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{i(t-1)}\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\delta_{\varepsilon e}\right] \\ \Rightarrow\ & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_iV_i + \varpi_{\varepsilon e}\left(\int W_i^2\right)+\delta_{\varepsilon e}\right] - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon e}\left(\int W_i^2\right)+\delta_{\varepsilon e}\right] \\ =\ & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_iV_i\right] \\ \Rightarrow\ & N\left(0, \frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^2}{6}\right). \end{aligned}$$

Also it is easy to see that

$$I = \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right] = \frac{1}{\sqrt{T}}O_p(1),$$

$$II = \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right) = \frac{1}{\sqrt{T}}o_p(1)\,O_p(1)$$

and

$$IV = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right) = o_p(1)\,O_p(1)$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. Hence, we have

$$\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it} - \sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{i(t-1)}\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\delta_{\varepsilon e}\right] \Rightarrow N\left(0, \frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^2}{6}\right),$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. ■

B Proof of Theorem 1

Proof. The proof is straightforward by using lemmas 1 and 2. ■

C Proof of Theorem 2

The following lemmas will be used to prove Theorem 2.

Lemma 3 *If Assumptions* $1-2$ *hold, then*

1. If $|\lambda|<1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$.
2. If $\lambda=1$, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{6}$.

Proof. Consider (1). For a fixed n, it is clear to see that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\\
=\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}^2\right]-\frac{1}{n}\sum_{i=1}^{n}\left[\overline{x}_i^2\right]\\
\xrightarrow{p}\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{\left(1-\lambda\right)^2\left(1+\lambda\right)}\right]\\
=\;&\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{\left(1-\lambda\right)^2\left(1+\lambda\right)}
\end{aligned}$$

as $T\to\infty$ because $\overline{x}_i=\frac{1}{T}\sum_{t=1}^{T}x_{it}\xrightarrow{p}E\left(x_{it}\right)=0$. Hence,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\xrightarrow{p}\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{\left(1-\lambda\right)^2\left(1+\lambda\right)}$$

holds for all n and hence it holds for a large n as well. This proves (1).

Next we consider (2). Note for a fixed n

$$\begin{aligned}
&\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\\
=\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\right]\\
\Rightarrow\;&\frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int\tilde{W}_i^2\right)
\end{aligned}$$

as $T\to\infty$. As $n\to\infty$,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int\tilde{W}_i^2\right)\xrightarrow{p}\frac{\varpi_\varepsilon^2}{6}.$$

by a LLN since

$$E\left(\int \tilde{W}_i^2\right) = \frac{1}{6}.$$

Hence,

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{6}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.This proves (2). ■

Lemma 4 *If Assumptions* $1-2$ *hold, then*

1. If $|\rho| < 1$ and $|\lambda| < 1$,

   (a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \xrightarrow{p} \lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)$

   (b) $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} - \sqrt{nT}\lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)$
   $\Rightarrow N\left(0, \frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{(1-\rho\lambda)^2}\right)$
   where $\psi_{0r} = E\left(\varepsilon_{t-r}^2 e_t^2\right)$, $\psi_{r0} = E\left(\varepsilon_t^2 e_{t-r}^2\right)$, and $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,

   (a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda}$

   (b) $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} - \sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}(\nu_{it}-\bar{\nu}_i)e_{it}\right)\frac{\varpi_{e\varepsilon}}{\varpi_e^2}+\delta_{e\varepsilon}}{1-\lambda}\right] \Rightarrow$
   $N\left(0, \frac{\varpi_{\varepsilon.e}\varpi_e^2}{6(1-\lambda)^2}\right)$.

3. If $|\rho| < 1$ and $\lambda = 1$,

   (a) $\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \Rightarrow \frac{1}{1-\rho}\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i dV_i\right) + \varpi_{\varepsilon e}\left(\int \widetilde{W}_i dW_i'\right) + \delta_{\varepsilon e}\right]$,

   (b) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{\varepsilon e}+\delta_{\varepsilon e}}{1-\rho}$,

   (c) $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} - \sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\delta_{\varepsilon e}}{1-\rho}\right] \Rightarrow$
   $N\left(0, \frac{\varpi_{e.\varepsilon}\varpi_\varepsilon^2}{6(1-\rho)^2}\right)$.

4. If $\rho = 1$ and $\lambda = 1$,

   (a) $\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \Rightarrow \varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right) + \varpi_{\varepsilon e}\left(\int \widetilde{W}_i^2\right) + \delta_{\varepsilon e}$,

   (b) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it}\right] \xrightarrow{p} \frac{\varpi_{\varepsilon e}}{6} + \delta_{\varepsilon e}$,

(c) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it}\right]-\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\delta_{\varepsilon e}\right]\Rightarrow N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^2}{90}\right)$.

Proof.

Consider (1). For part (a), note that

$$\begin{aligned} & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} \\ = \; & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]-\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right] \end{aligned}$$

Because

$$\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]=\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)+o_p\left(1\right)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. Also it is easy to see that

$$\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right]=o_p\left(1\right).$$

Hence, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}=\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)+o_p\left(1\right).$$

This proves (a).

For part (b), for a fixed n,

$$\begin{aligned} & \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} \\ = \; & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right] \end{aligned}$$

Because

$$\begin{aligned} & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\ \Rightarrow \; & N\left(0,\frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{\left(1-\rho\lambda\right)^2}\right) \end{aligned}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$, where $\psi_{0r} = E\left(\varepsilon_{t-r}^2 e_t^2\right)$, $\psi_{r0} = E\left(\varepsilon_t^2 e_{t-r}^2\right)$, $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$. Also it is easy to see that

$$\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right] = o_p(1)$$

Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} - \sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\\
= &\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right) - \sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\right]\\
&-\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right]\\
\Rightarrow &\ N\left(0, \frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{\left(1-\rho\lambda\right)^2}\right).
\end{aligned}$$

Consider (2). For part (a), for a fixed n, note that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\\
= &\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\left(\nu_{it}-\overline{\nu}_i\right)\right]\\
\Rightarrow &\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\varpi_e\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_i dW_i\right)+\varpi_{e\varepsilon}\left(\int\widetilde{V}_i dV_i\right)+\delta_{e\varepsilon}}{1-\lambda}\right]
\end{aligned}$$

as $T\to\infty$, by a CLT because

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\varpi_e\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_i dW_i\right)+\varpi_{e\varepsilon}\left(\int\widetilde{V}_i dV_i\right)+\delta_{e\varepsilon}}{1-\lambda}\right]\\
&\overset{p}{\to}\frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda},
\end{aligned}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. Hence, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} \overset{p}{\to} \frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

For part (b), for a fixed n,

$$\begin{aligned}
& \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it} \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\left(\nu_{it}-\overline{\nu}_{i}\right)\right] \\
\Rightarrow & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{e}\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_{i}dW_{i}\right)+\varpi_{e\varepsilon}\left(\int\widetilde{V}_{i}dV_{i}\right)+\delta_{e\varepsilon}}{1-\lambda}\right]
\end{aligned}$$

as $T\to\infty$, by a CLT. Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it}-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(\nu_{it}-\bar{\nu}_{i}\right)e_{it}\right)\frac{\varpi_{e\varepsilon}}{\varpi_{e}^{2}}+\delta_{e\varepsilon}}{1-\lambda}\right] \\
\Rightarrow & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{e}\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_{i}dW_{i}\right)+\varpi_{e\varepsilon}\left(\int\widetilde{V}_{i}dV_{i}\right)+\delta_{e\varepsilon}}{1-\lambda}\right]-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{e\varepsilon}\left(\int\widetilde{V}_{i}dV_{i}\right)+\delta_{e\varepsilon}}{1-\lambda}\right] \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{e}\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_{i}dW_{i}\right)}{1-\lambda}\right] \\
\Rightarrow & N\left(0,\frac{\varpi_{\varepsilon.e}\varpi_{e}^{2}}{6\left(1-\lambda\right)^{2}}\right).
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Consider (3). For part (a), note that for a fixed n,

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it} \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it}\right] \\
\Rightarrow & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_{i}dV_{i}\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_{i}dW_{i}\right)+\delta_{\varepsilon e}}{1-\rho}\right]
\end{aligned}$$

as $T\to\infty$, by a central limit theorem. And

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_{i}dV_{i}\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_{i}dW_{i}\right)+\delta_{\varepsilon e}}{1-\rho}\right] \\
& \overset{p}{\to}\frac{-\frac{1}{2}\varpi_{\varepsilon e}+\delta_{\varepsilon e}}{1-\rho},
\end{aligned}$$

as $n \to \infty$. Hence, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\xrightarrow{p}\frac{-\frac{1}{2}\varpi_{\varepsilon e}+\delta_{\varepsilon e}}{1-\rho}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), for a fixed n,

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\right]\\
\Rightarrow\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_i dW_i\right)+\delta_{\varepsilon e}}{1-\rho}\right],
\end{aligned}$$

as $T\to\infty$, by a central limit theorem. Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^{2}}+\delta_{\varepsilon e}}{1-\rho}\right]\\
\Rightarrow\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_i dW_i\right)+\delta_{\varepsilon e}}{1-\rho}\right]\\
&-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon e}\left(\int\widetilde{W}_i dW_i\right)+\delta_{\varepsilon e}}{1-\rho}\right]\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_i dV_i\right)}{1-\rho}\right]\\
\Rightarrow\;&N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^{2}}{6\left(1-\rho\right)^{2}}\right).
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

If $\rho=1$ and $\lambda=1$,

Consider (4). For part (a), note that

$$\begin{aligned}
& \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\,\nu_{it} \\
= \; & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\,\nu_{it}\right] \\
\Rightarrow \; & \frac{1}{n}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W_i}^2\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right]
\end{aligned}$$

as $T \to \infty$, by a central limit theorem. And

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W_i}^2\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right] \\
& \xrightarrow{p} \frac{\varpi_{\varepsilon e}}{6}+\delta_{\varepsilon e}
\end{aligned}$$

as $n \to \infty$. Hence, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\,\nu_{it} \xrightarrow{p} \frac{\varpi_{\varepsilon e}}{6}+\delta_{\varepsilon e}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

For part (b), for a fixed n,

$$\begin{aligned}
& \frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\,\nu_{it} \\
= \; & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\,\nu_{it}\right] \\
\Rightarrow \; & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W_i}^2\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right],
\end{aligned}$$

as $T \to \infty$, by a central limit theorem. Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\,\nu_{it}-\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)^2\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\delta_{\varepsilon e}\right] \\
\Rightarrow \; & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W_i}^2\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right]-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\int \widetilde{W_i}^2\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right] \\
= \; & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)\right] \\
\Rightarrow \; & N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^2}{90}\right).
\end{aligned}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. ■

D Proof of Theorem 2

Proof. Te proof of Theorem 2 is straightforward with above lemmas. ■

E Proof of Theorem 3

The following lemmas will be used to prove Theorem 3.

Lemma 5 *If Assumptions* $1-2$ *hold, then*

1. If $|\lambda|<1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2 \overset{p}{\to} \frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}$,
2. If $\lambda=1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2 \overset{p}{\to} \sigma_\varepsilon^2$.

Proof. Consider (1). If $|\lambda|<1$, for a fixed n, it is clear to see that

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2 \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2\right] \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}^2+\frac{1}{T}\sum_{t=1}^{T}x_{it-1}^2-\frac{2}{T}\sum_{t=1}^{T}x_{it}x_{it-1}\right] \\
& \overset{p}{\to} \frac{1}{n}\sum_{i=1}^{n}\left[2\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)-2\left[\lambda\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)+\frac{\gamma_\varepsilon^2}{1-\lambda}\right]\right] \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right] \\
& \overset{p}{\to} \frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}
\end{aligned}$$

as $T\to\infty$ because $\frac{1}{T}\sum_{t=1}^{T}x_{it}^2 \overset{p}{\to} E\left(x_{it}^2\right)=\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}$ and $\frac{1}{T}\sum_{t=1}^{T}x_{it}x_{it-1} \overset{p}{\to} E\left(x_{it}x_{it-1}\right)=\lambda\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)+\frac{\gamma_\varepsilon^2}{1-\lambda}$. Hence,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \overset{p}{\to} \frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}$$

holds for all n and hence it holds for a large n as well. This proves (1).

Next we consider (2). If $\lambda = 1$,

$$\begin{aligned} & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^{2} \\ = & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^{2}\right] \\ = & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{2}\right]\xrightarrow{p}\sigma_{\varepsilon}^{2} \end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$.This proves (2). ■.

Lemma 6 *If Assumptions* $1-2$ *hold, then*

1. If $|\rho|<1$ and $|\lambda|<1$,

   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)$

   (b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)$ $\Rightarrow N\left(0,\frac{(2-\lambda-\rho)^{2}\psi_{00}+\sum_{r=1}^{\infty}\left(-\rho^{r-1}+2\rho^{r}-\rho^{r+1}\right)^{2}\psi_{0r}+\sum_{r=1}^{\infty}\left(-\lambda^{r-1}+2\lambda^{r}-\lambda^{r+1}\right)^{2}\psi_{r0}}{(1-\rho\lambda)^{2}}\right)$ where $\psi_{0r}=E\left(\varepsilon_{t-r}^{2}e_{t}^{2}\right)$, $\psi_{r0}=E\left(\varepsilon_{t}^{2}e_{t-r}^{2}\right)$, $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right)$.

2. If $\rho=1$ and $|\lambda|<1$,

   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+(\lambda-1)x_{it-1}\right)e_{it}$

   (b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+(\lambda-1)x_{it-1}\right)e_{it}\Rightarrow$ $N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)$, where $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right)$

3. If $|\rho|<1$ and $\lambda=1$,

   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)$

   (b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)\Rightarrow$ $N\left(0,\frac{2\psi_{00}}{1+\rho}\right)$, where $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right)$.

4. If $\rho=1$ and $\lambda=1$,

   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]\xrightarrow{p}\sigma_{\varepsilon e}$

(b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{nT}\sigma_{\varepsilon e}\Rightarrow N\left(0,\varpi_{e}^{2}\varpi_{\varepsilon}^{2}\right).$

Proof. Consider (1). For part (a), it is easy to see that

$$\begin{aligned}&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\&\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), note that

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+(\lambda-1)x_{it-1}\right)\left(e_{it}+(\rho-1)\nu_{it-1}\right)\right]\\=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+(\lambda-1)\varepsilon_{it-1}+\lambda(\lambda-1)\varepsilon_{it-2}+\lambda^{2}(\lambda-1)\varepsilon_{it-3}+\cdots\right)\\&\left(e_{it}+(\rho-1)e_{it-1}+\rho(\rho-1)e_{it-2}+\rho^{2}(\rho-1)e_{it-3}+\cdots\right)]\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+(\lambda-1)\varepsilon_{it-1}+\lambda(\lambda-1)\varepsilon_{it-2}+\lambda^{2}(\lambda-1)\varepsilon_{it-3}+\cdots\right)\\&\left(e_{it}+(\rho-1)e_{it-1}+\rho(\rho-1)e_{it-2}+\rho^{2}(\rho-1)e_{it-3}+\cdots\right)]\\&-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\\Rightarrow&N\left(0,\frac{(2-\lambda-\rho)^{2}\psi_{00}+\sum_{r=1}^{\infty}\left(-\rho^{r-1}+2\rho^{r}-\rho^{r+1}\right)^{2}\psi_{0r}+\sum_{r=1}^{\infty}\left(-\lambda^{r-1}+2\lambda^{r}-\lambda^{r+1}\right)^{2}\psi_{r0}}{(1-\rho\lambda)^{2}}\right)\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$, where $\psi_{0r}=E\left(\varepsilon_{t-r}^{2}e_{t}^{2}\right)$, $\psi_{r0}=E\left(\varepsilon_{t}^{2}e_{t-r}^{2}\right)$, $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right).$

Consider (2). For part (a), it is easy to see that

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}\right] \\
& \xrightarrow{p} \lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$.

For part (b), using Lemma A0 in Choi (1999), we have

$$\begin{aligned}
& \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it} \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it} \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left[\left(\varepsilon_{it}+\left(\lambda-1\right)\varepsilon_{it-1}+\lambda\left(\lambda-1\right)\varepsilon_{it-2}+\lambda^{2}\left(\lambda-1\right)\varepsilon_{it-3}+\cdots\right)e_{it}\right]\right] \\
& -\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it} \\
\Rightarrow & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)\right]=N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$, where $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right)$.

Consider (3). For part (a), it is easy to see that

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)\right] \\
& \xrightarrow{p} \lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$.

For part (b),

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\left(e_{it}+\left(\rho-1\right)\nu_{it-1}\right)\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left[\varepsilon_{it}\left(e_{it}+\left(\rho-1\right)e_{it-1}+\rho\left(\rho-1\right)e_{it-2}+\rho^{2}\left(\rho-1\right)e_{it-3}+\cdots\right)\right]\right] \\
& -\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
\Rightarrow & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[N\left(0,\frac{2\psi_{00}}{1+\rho}\right)\right]=N\left(0,\frac{2\psi_{00}}{1+\rho}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$, where $\psi_{00}=E\left(\varepsilon_t^2 e_t^2\right)$.

Consider (4). For part (a),

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}e_{it}\right] \\
& \overset{p}{\to}\sigma_{\varepsilon e}
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), note that

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it}\right]
\end{aligned}$$

For a fixed n,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - x_{it-1}\right)\left(\nu_{it} - \nu_{it-1}\right) - \sqrt{nT}\sigma_{\varepsilon e} \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_{it} e_{it} \right] - \sqrt{nT}\sigma_{\varepsilon e} \\
\Rightarrow & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i
\end{aligned}$$

as $T \to \infty$, by a central limit theorem, where $Z_i \sim N\left(0, \varpi_e^2 \varpi_\varepsilon^2\right)$.

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \Rightarrow N\left(0, \varpi_e^2 \varpi_\varepsilon^2\right)$$

as $n \to \infty$. Hence, we have

$$\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - x_{it-1}\right)\left(\nu_{it} - \nu_{it-1}\right) - \sqrt{nT}\sigma_{\varepsilon e} \Rightarrow N\left(0, \varpi_e^2 \varpi_\varepsilon^2\right)$$

as $(n, T) \overset{\text{seq}}{\to} \infty$. ■

Proof of Theorem 3:

Proof. By $\widehat{\beta}_{FD} - \beta = \frac{\sum_{i=1}^{n} \sum_{t=1}^{T} (x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1})}{\sum_{i=1}^{n} \sum_{t=1}^{T} (x_{it} - x_{it-1})^2}$, the proof of Theorem 3 is straightforward with above lemmas. ■

F Proof of Theorem 4

Define $\mathbf{z} = [\boldsymbol{\iota}_{nT}, \mathbf{x}]$, then

$$\begin{aligned}
\begin{pmatrix} \hat{\alpha}_{GLS} \\ \widehat{\beta}_{GLS} \end{pmatrix} &= \left(\mathbf{z}'\Phi^{-1}\mathbf{z}\right)^{-1} \left(\mathbf{z}'\Phi^{-1}\mathbf{y}\right) \\
&= \left(\begin{bmatrix} \boldsymbol{\iota}_{nT}' \\ \mathbf{x}' \end{bmatrix} \Phi^{-1} \begin{bmatrix} \boldsymbol{\iota}_{nT} & \mathbf{x} \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} \boldsymbol{\iota}_{nT}' \\ \mathbf{x}' \end{bmatrix} \Phi^{-1}\mathbf{y} \right) \\
&= \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} & \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \\ \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} & \mathbf{x}'\Phi^{-1}\mathbf{x} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} \\ \mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} \\ \mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{12}\mathbf{x}'\Phi^{-1}\mathbf{y} \\ F_{21}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{22}\mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix}
\end{aligned}$$

where

$$
\begin{aligned}
F_{11} &= \left[\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} - \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\left(\mathbf{x}'\Phi^{-1}\mathbf{x}\right)^{-1}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right]^{-1}, \\
F_{12} &= -\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}, \\
F_{21} &= -\left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}, \\
F_{22} &= \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\widehat{\beta}_{GLS} &= F_{21}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{22}'\mathbf{x}'\Phi^{-1}\mathbf{y} \\
&= \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}\left[\mathbf{x}'\Phi^{-1}\mathbf{y} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y}\right]
\end{aligned}
$$

and

$$\widehat{\beta}_{GLS} - \beta = G_1^{-1}G_2,$$

where $G_1 = \mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{x}$ and $G_2 = \mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{u}$.

With the definition of Φ, we have

$$\mathbf{x}'\Phi^{-1}\mathbf{x} = \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}_i^{-1}\mathbf{x}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}_i^{-1}\mathbf{x}_i\right)$$

$$
\begin{aligned}
\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}_i^{-1}\mathbf{x}_i\right) \\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\theta\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) \\
&= \frac{1}{\varpi_e^2+\theta\sigma_\mu^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right)\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\theta - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\theta^2\right)\\
&= \frac{1}{\varpi_e^2}n\theta\left(1-\frac{\theta\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\right)\\
&= \frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}'\Phi^{-1}\mathbf{u} &= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{u}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'A^{-1}\mathbf{u}_i\right)\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left[\left(\mu_i\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\mu_i\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) + \left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-}\right.\right.\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left[\left(\mu_i\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\mu_i\theta\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) + \left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left[\frac{\mu_i\sigma_e^2}{\sigma_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T + \left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} &= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\left(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{u}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{u}_i\right)\\
&= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\left[\left(\mu_i\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\mu_i\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) + \left(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}\right.\right.\\
&= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\left[\left(\mu_i\theta - \frac{\mu_i\theta^2\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\right) + \left(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\theta\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]\\
&= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\left[\frac{\mu_i\theta\varpi_e^2}{\varpi_e^2+\theta\sigma_\mu^2} + \frac{\theta\varpi_e^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right]\\
&= \sum_{i=1}^{n}\left[\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i+\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]
\end{aligned}
$$

because $\mathbf{u}_i = \mu_i\boldsymbol{\iota}_T + \boldsymbol{\nu}_i$.

When $|\rho| < 1$, $\mathbf{A} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1}\\ \rho & 1 & \rho & \cdots & \rho^{T-2}\\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1\end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -\rho & 0 & 0 & \cdots & 0\\ -\rho & 1+\rho^2 & -\rho & 0 & \cdots & 0\\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -\rho & 1+\\ 0 & 0 & 0 & \cdots & 0 & -\rho \end{bmatrix}$

It can be shown $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$, where $\mathbf{C} = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -\rho & 1 & 0 \\ 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix}$ is the Prais-Winsten (PW) transformation matrix suggested in Baltagi and Li (1991). Hence

$$\mathbf{C}\mathbf{x}_i = \begin{bmatrix} \sqrt{1-\rho^2}x'_{i1} \\ x'_{i2} - \rho x'_{i1} \\ x'_{i3} - \rho x'_{i2} \\ \vdots \\ x'_{iT-1} - \rho x'_{iT-2} \\ x'_{iT} - \rho x'_{iT-1} \end{bmatrix}, \ \mathbf{C}\boldsymbol{\nu}_i = \begin{bmatrix} \sqrt{1-\rho^2}\nu'_{i1} \\ \nu'_{i2} - \rho \nu'_{i1} \\ \nu'_{i3} - \rho \nu'_{i2} \\ \vdots \\ \nu'_{iT-1} - \rho \nu'_{iT-2} \\ \nu'_{iT} - \rho \nu'_{iT-1} \end{bmatrix} = \begin{bmatrix} \sqrt{1-\rho^2}e'_{i1} \\ e'_{i2} \\ e'_{i3} \\ \vdots \\ e'_{iT-1} \\ e'_{iT} \end{bmatrix},$$

$\mathbf{C}\boldsymbol{\iota}_T = \begin{bmatrix} \sqrt{1-\rho^2} \\ 1-\rho \\ 1-\rho \\ \vdots \\ 1-\rho \\ 1-\rho \end{bmatrix}$. And

$$\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i = \mathbf{x}'_i\mathbf{C}'\mathbf{C}\mathbf{x}_i \approx \sum_{t=1}^{T} \left(x_{it} - \rho x_{it-1}\right)^2$$

$$\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_i = \mathbf{x}'_i\mathbf{C}'\mathbf{C}\boldsymbol{\nu}_i \approx \sum_{t=1}^{T} \left(x_{it} - \rho x_{it-1}\right) e_{it}$$

$$\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T = \mathbf{x}'_i\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T \approx (1-\rho)\sum_{t=1}^{T} \left(x_{it} - \rho x_{it-1}\right)$$

$$\boldsymbol{\iota}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i = \boldsymbol{\iota}'_T\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T\boldsymbol{\nu}_i \approx (1-\rho)\sum_{t=1}^{T} e_{it}$$

$$\theta = \boldsymbol{\iota}'_T\mathbf{A}^{-1}\boldsymbol{\iota}_T = \boldsymbol{\iota}'_T\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T \approx \sum_{t=1}^{T} (1-\rho)^2 = O\left((1-\rho)^2 T\right).$$

When $\rho = 1$, $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & T \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$.

It can be shown $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$, where $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$.

Hence

$$\mathbf{C}\mathbf{x}_i = \begin{bmatrix} x'_{i1} \\ x'_{i2} - x'_{i1} \\ x'_{i3} - x'_{i2} \\ \vdots \\ x'_{iT-1} - x'_{iT-2} \\ x'_{iT} - x'_{iT-1} \end{bmatrix}, \mathbf{C}\boldsymbol{\nu}_i = \begin{bmatrix} \nu'_{i1} \\ \nu'_{i2} - \nu'_{i1} \\ \nu'_{i3} - \nu'_{i2} \\ \vdots \\ \nu'_{iT-1} - \nu'_{iT-2} \\ \nu'_{iT} - \nu'_{iT-1} \end{bmatrix} = \begin{bmatrix} e'_{i1} \\ e'_{i2} \\ e'_{i3} \\ \vdots \\ e'_{iT-1} \\ e'_{iT} \end{bmatrix}, \mathbf{C}\boldsymbol{\iota}_T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

And

$$\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i = \mathbf{x}_i'\mathbf{C}'\mathbf{C}\mathbf{x}_i \approx \sum_{t=1}^{T} (x_{it} - x_{it-1})^2$$

$$\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \mathbf{x}_i'\mathbf{C}'\mathbf{C}\boldsymbol{\nu}_i \approx \sum_{t=1}^{T} (x_{it} - x_{it-1})\, e_{it}$$

$$\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \mathbf{x}_i'\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T = x'_{i1}$$

$$\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \boldsymbol{\iota}_T'\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T\boldsymbol{\nu}_i = \nu'_{i1}$$

$$\theta = \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \boldsymbol{\iota}_T'\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T = 1.$$

The following lemmas will be used to prove Theorem 4.

Lemma 7 *If Assumptions* $1-2$ *hold, then*

1. If $|\rho| < 1$ and $|\lambda| < 1$,

   (a) $\frac{1}{nT}G_1 \xrightarrow{p} \frac{1}{\varpi_e^2}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right]$,

   (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]$,

(c) $\frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right] \Rightarrow \frac{1}{\varpi_e^2}N\left(0, \frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,

   (a) $\frac{1}{nT}G_1 \xrightarrow{p} \frac{1}{\varpi_e^2}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right]$,

   (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right]$,

   (c) $\frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \Rightarrow \frac{1}{\varpi_e^2}N\left(0, \frac{2\psi_{00}}{1+\lambda}\right)$.

3. If $|\rho| < 1$ and $\lambda = 1$,

   (a) $\frac{1}{nT^2}G_1 \xrightarrow{p} \frac{(1-\rho)^2\varpi_\varepsilon^2}{6\varpi_e^2}$,

   (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{1}{\varpi_e^2}\left[(1-\rho)\left[-\frac{1}{2}\varpi_{\varepsilon e} + \gamma_{\varepsilon e}\right] + \sigma_{\varepsilon e}\right]$,

   (c) $\frac{1}{\sqrt{nT}}G_2 - \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\bar{x}_i\right)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2} + \gamma_{\varepsilon e} + \frac{\sigma_{\varepsilon e}}{1-\rho}\right] \Rightarrow$
   $N\left(0, \frac{(1-\rho)^2\varpi_\varepsilon^2\varpi_{e.\varepsilon}}{6\varpi_e^4}\right)$,

4. If $\rho = 1$ and $\lambda = 1$,

   (a) $\frac{1}{nT}G_1 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2}$,

   (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{\sigma_{\varepsilon e}}{\varpi_e^2}$,

   (c) $\frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2} \Rightarrow \frac{1}{\varpi_e^2}N\left(0, \varpi_e^2\varpi_\varepsilon^2\right)$.

Proof.

Note that

$$\frac{1}{nT}G_1 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n}$$

First consider

$$\begin{aligned} & \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\ = \; & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{T}\right) \end{aligned}$$

For a fixed n,

$$\begin{aligned}
&\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i \\
=\ & \frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)^2 \\
=\ & \frac{1}{T}\sum_{t=1}^{T}x_{it}^2+\rho^2\frac{1}{T}\sum_{t=1}^{T}x_{it-1}^2-\rho\frac{1}{T}\sum_{t=1}^{T}2x_{it-1}x_{it} \\
&\xrightarrow{p}\left(1+\rho^2\right)\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)-2\rho\left[\lambda\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)+\frac{\gamma_\varepsilon^2}{1-\lambda}\right] \\
=\ & \frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}
\end{aligned}$$

by a CLT and also

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T=(1-\rho)\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)\xrightarrow{p}0$$

as $T\to\infty$. So, with $\theta=O\left((1-\rho)^2T\right)$ when $|\rho|<1$,

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
=\ & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i-\frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{T}\right) \\
&\xrightarrow{p}\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right] \\
=\ & \frac{1}{\varpi_e^2}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right]
\end{aligned}$$

holds for all n and hence it holds for a large n as well.

Also because

$$\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_e^2+\theta\sigma_\mu^2}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right)\xrightarrow{p}0$$

because $\theta=O(T)$ and $\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\xrightarrow{p}0$. And

$$\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}=\frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned}\frac{1}{nT}G_1 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x}-\frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n}\\ &\xrightarrow{p}\frac{1}{\varpi_e^2}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right]\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Note that

$$\frac{1}{nT}G_2=\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}$$

First consider

$$\begin{aligned}&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}\\ =\;&\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}\right)\right].\end{aligned}$$

Because $\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\xrightarrow{p}0$, which has been proved in part (a), and

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i=\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)e_{it}\xrightarrow{p}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right],$$

and

$$\frac{1}{T}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i=(1-\rho)\frac{1}{T}\sum_{t=1}^{T}e_{it}\xrightarrow{p}0$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. Hence,

$$\begin{aligned}&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}\\ =\;&\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}\right)\right]\\ &\xrightarrow{p}\frac{1}{T}o(1)+\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\\ =\;&\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Also because

$$\frac{1}{nT}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}=\frac{1}{n}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\left(\frac{\mu_i}{T}+\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}\right)\xrightarrow{p}0$$

and $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$ and $\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}\frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}\\ &\xrightarrow{p}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$.

Note that

$$\frac{1}{\sqrt{nT}}G_2=\frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{u}}{\sqrt{nT}}$$

First consider

$$\begin{aligned}&\sqrt{nT}\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}\\ =&\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}-\frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]\end{aligned}$$

For a fixed n, $\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\xrightarrow{p}0$, which has been proved in part (a), and

$$\begin{aligned}&\sqrt{T}\left[\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right]\\ =&\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)e_{it}-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right]\\ =&\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+\left(\lambda-\rho\right)x_{it-1}\right)e_{it}-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right]\\ =&\sqrt{T}\{\frac{1}{T}\sum_{t=1}^{T}\left[\left(\varepsilon_{it}+\left(\lambda-\rho\right)\varepsilon_{it-1}+\lambda\left(\lambda-\rho\right)\varepsilon_{it-2}+\lambda^2\left(\lambda-\rho\right)\varepsilon_{it-3}+\cdots\right)e_{it}\right]\\ &-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\}\\ =&N\left(0,\frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right)\end{aligned}$$

where $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$ and

$$\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = (1-\rho)\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_{it} \Rightarrow (1-\rho)\varpi_e N(0,1)$$

as $T \to \infty$. Hence,

$$\begin{aligned}
&\sqrt{nT}\left[\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right] \\
= &\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right) + \left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} - \frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
&-\frac{1}{\varpi_e^2}\sqrt{nT}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right] \\
\Rightarrow &\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}N\left(0, \frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right) \\
= &\frac{1}{\varpi_e^2}N\left(0, \frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right)
\end{aligned}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

Also consider that

$$\begin{aligned}
\frac{1}{\sqrt{nT}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\left(\mu_i + \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
&= \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\mu_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}
\end{aligned}$$

It is easy to see that

$$\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\mu_i = \frac{1}{\sqrt{T}}o_p(1)$$

Also, for a fixed n,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} \Rightarrow \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{(1-\rho)\varpi_e}{\sigma_\mu^2}N(0,1)\right]$$

as $T \to \infty$ because $\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i \Rightarrow (1-\rho)\varpi_e N(0,1)$, which has been proved in part (e). Also,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{(1-\rho)\varpi_e}{\sigma_\mu^2}N(0,1)\right] \Rightarrow \frac{(1-\rho)\varpi_e}{\sigma_\mu^2}N(0,1)$$

as $n \to \infty$. Therefore,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \frac{\boldsymbol{\iota}_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \Rightarrow \frac{(1-\rho)\,\varpi_e}{\sigma_\mu^2} N(0,1)$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. So,

$$\frac{1}{\sqrt{nT}} \boldsymbol{\iota}_{nT}' \Phi^{-1} \mathbf{u} \Rightarrow \frac{(1-\rho)\,\varpi_e}{\sigma_\mu^2} N(0,1)$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} E\left[(x_{it} - \rho x_{it-1})\, e_{it}\right] \\ = \; & \frac{1}{\sqrt{nT}} \mathbf{x}' \Phi^{-1} \mathbf{u} - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} E\left[(x_{it} - \rho x_{it-1})\, e_{it}\right] \\ & - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \left(\frac{\boldsymbol{\iota}_{nT}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}_{NT}' \Phi^{-1} \mathbf{u}}{\sqrt{nT}} \\ \Rightarrow \; & \frac{1}{\sigma_e^2} N\left(0, \frac{\left(1 - 2\rho\lambda + \rho^2\right) \psi_{00}}{1 - \lambda^2}\right). \end{aligned}$$

Note that

$$\frac{1}{nT} G_1 = \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \left(\frac{\boldsymbol{\iota}_{nT}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}_{nT}' \Phi^{-1} \mathbf{x}}{n}$$

First consider that

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} \\ = \; & \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\mathbf{x}_i' \mathbf{A}^{-1} \mathbf{x}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}_i' \mathbf{A}^{-1} \boldsymbol{\iota}_T}{\sqrt{T}} \frac{\boldsymbol{\iota}_T' \mathbf{A}^{-1} \mathbf{x}_i}{\sqrt{T}} \right) \end{aligned}$$

for a fixed n,

$$\begin{aligned} & \frac{1}{T} \mathbf{x}_i' \mathbf{A}^{-1} \mathbf{x}_i \\ = \; & \frac{1}{T} \sum_{t=1}^{T} \left(x_{it} - x_{it-1}\right)^2 \\ = \; & \frac{1}{T} \sum_{t=1}^{T} x_{it}^2 + \frac{1}{T} \sum_{t=1}^{T} x_{it-1}^2 - \frac{1}{T} \sum_{t=1}^{T} 2 x_{it-1} x_{it} \\ & \overset{p}{\to} 2 \left(\frac{\sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2\lambda \gamma_\varepsilon^2}{1 - \lambda} \right) - 2 \left[\lambda \left(\frac{\sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2\lambda \gamma_\varepsilon^2}{1 - \lambda} \right) + \frac{\gamma_\varepsilon^2}{1 - \lambda} \right] \\ = \; & \frac{2\sigma_\varepsilon^2}{1 + \lambda} + \frac{2\left(-2\lambda^2 + 2\lambda - 1\right) \gamma_\varepsilon^2}{1 - \lambda} \end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0,$$

as $T \to \infty$. Hence

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= \;& \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{\sqrt{T}}\right) \\
&\xrightarrow{p} \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right] \\
= \;& \frac{1}{\varpi_e^2}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right]
\end{aligned}$$

holds for all n and hence it holds for a large n as well.

Also because that

$$\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_e^2+\theta\sigma_\mu^2}\left(X_i'A^{-1}\iota_T\right) \xrightarrow{p} 0$$

since $\theta = 1$, and

$$\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_1 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n} \\
&\xrightarrow{p} \frac{1}{\varpi_e^2}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right]
\end{aligned}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}$$

First consider that

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \;& \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right) + \left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]
\end{aligned}$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \xrightarrow{p} 0$ which is proved in part (a), and

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{T}\sum_{t=1}^{T}(x_{it}-x_{it-1})\,e_{it} \xrightarrow{p} \lim \frac{1}{T}\sum_{t=1}^{T} E\left[(x_{it}-x_{it-1})\,e_{it}\right],$$

and

$$\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}\nu_{i1}' \xrightarrow{p} 0,$$

as $T \to \infty$. Hence,

$$\begin{aligned} & \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\ = & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\ & \xrightarrow{p} \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[(x_{it}-x_{it-1})\,e_{it}\right] \end{aligned}$$

as $n \to \infty$.

Also because that

$$\begin{aligned} & \frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\ = & \frac{1}{n}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i+\iota_T'A^{-1}\nu_i\right) \\ = & \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}\left(\mu_i+\nu_{i1}'\right) \\ & \xrightarrow{p} 0 \end{aligned}$$

and $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned} \frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT} \\ & \xrightarrow{p} \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[(x_{it}-x_{it-1})\,e_{it}\right] \end{aligned}$$

as $(n,T) \xrightarrow{\text{seq}} \infty$.

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}$$

First consider that

$$
\begin{aligned}
& \sqrt{nT}\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}-\frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]
\end{aligned}
$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \xrightarrow{p} 0$ which is proved in part (a), and

$$
\begin{aligned}
& \sqrt{T}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right]\right) \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)e_{it}-\sqrt{T}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}-\sqrt{T}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\left(\varepsilon_{it}+\left(\lambda-1\right)\varepsilon_{it-1}+\lambda\left(\lambda-1\right)\varepsilon_{it-2}+\lambda^2\left(\lambda-1\right)\varepsilon_{it-3}+\cdots\right)e_{it}\right] \\
& -\sqrt{T}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
\Rightarrow \quad & N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)
\end{aligned}
$$

where $\psi_{00}=E\left(\varepsilon_t^2e_t^2\right)$, and

$$
\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i=\frac{1}{\sqrt{T}}x_{i1}\nu_{i1}'\xrightarrow{p}0,
$$

as $T \to \infty$. Hence,

$$\begin{aligned}
& \sqrt{nT}\left(\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right]\right) \\
= & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right) \\
& +\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
& -\sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
\Rightarrow & o_p(1) + \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[N\left(0,\frac{2\psi_{00}}{1+\lambda}\right) - o_p(1)\right] \\
\Rightarrow & \frac{1}{\varpi_e^2}N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Also because that

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i+\iota_T'A^{-1}\nu_i\right) \\
\Rightarrow & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}\left(\mu_i+\nu_{i1}'\right) \\
\Rightarrow & \frac{1}{\varpi_e^2+\sigma_\mu^2}N\left(0,1\right)
\end{aligned}$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\overset{p}{\to}0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\overset{p}{\to}\frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
= & \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right] \\
& -\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{\sqrt{nT}} \\
\Rightarrow & \frac{1}{\varpi_e^2}N\left(0,\frac{2\psi_{00}}{1+\lambda}\right).
\end{aligned}$$

Note that

$$\frac{1}{nT^2}G_1 = \frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}}{n\sqrt{T}}$$

First consider that

$$\frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} = \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T^2}\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}'_T\mathbf{A}^{-1}\mathbf{x}_i}{T\sqrt{T}}\right)$$

For a fixed n, because $\theta = O\left((1-\rho)^2 T\right)$ when $|\rho| < 1$ and

$$\begin{aligned}
&\frac{1}{T^2}\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i \\
=\;& \frac{1}{T^2}\sum_{t=1}^{T}(x_{it}-\rho x_{it-1})^2 \\
=\;& \frac{1}{T^2}\sum_{t=1}^{T}((1-\rho)x_{it-1}+\varepsilon_{it})^2 \\
=\;& (1-\rho)^2\frac{1}{T^2}\sum_{t=1}^{T}x_{it-1}^2 + \frac{1}{T}\left[(1-\rho)\frac{1}{T}\sum_{t=1}^{T}2x_{it-1}\varepsilon_{it} + \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2\right] \\
\Rightarrow\;& (1-\rho)^2\varpi_\varepsilon^2\int W_i^2 + \frac{1}{T}o_p(1),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{T\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T \\
=\;& (1-\rho)\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}(x_{it}-\rho x_{it-1}) \\
=\;& (1-\rho)\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}((1-\rho)x_{it-1}+\varepsilon_{it}) \\
=\;& (1-\rho)^2\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}x_{it-1} + \frac{1}{\sqrt{T}}\left[(1-\rho)\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\right] \\
\Rightarrow\;& (1-\rho)^2\varpi_\varepsilon\int W_i + \frac{1}{\sqrt{T}}o_p(1),
\end{aligned}$$

as $T \to \infty$. Hence

$$\begin{aligned} \frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} &= \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{T\sqrt{T}}\right) \\ &\xrightarrow{p} \frac{1}{\varpi_e^2}E\left[(1-\rho)^2\varpi_\varepsilon^2\int W_i^2 - (1-\rho)^2\varpi_\varepsilon^2\left(\int W_i\right)^2\right] \\ &= \frac{(1-\rho)^2\varpi_\varepsilon^2}{6\varpi_e^2} \end{aligned}$$

as $n \to \infty$.

And because

$$\begin{aligned} \frac{1}{n\sqrt{T}}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{n}\sum_{i=1}^{n}\frac{T}{\varpi_e^2+\theta\sigma_\mu^2}\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right) \\ &\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sigma_\mu^2}(1-\rho)^2\varpi_\varepsilon\int W_i\right] \\ &\xrightarrow{p} 0 \end{aligned}$$

because $\theta = O(T)$ and $\frac{1}{T\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \Rightarrow (1-\rho)^2\varpi_\varepsilon\int W_i + o_p(1)$, which is proved in (a). And because

$$\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned} \frac{1}{nT^2}G_1 &= \frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n\sqrt{T}} \\ &\xrightarrow{p} \frac{(1-\rho)^2\varpi_\varepsilon^2}{6\varpi_e^2} \end{aligned}$$

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n\sqrt{T}}$$

First consider that

$$\begin{aligned} &\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\ = &\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right) + \left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \end{aligned}$$

For a fixed n, because $\frac{1}{T\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \Rightarrow (1-\rho)^2 \varpi_\varepsilon \int W_i + o_p(1)$, which is proved in (a) and

$$\begin{aligned}
& \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i \\
= \quad & \frac{1}{T}\sum_{t=1}^{T}(x_{it}-\rho x_{it-1})\,e_{it} \\
= \quad & \frac{1}{T}\sum_{t=1}^{T}((1-\rho)\,x_{it-1}+\varepsilon_{it})\,e_{it} \\
= \quad & (1-\rho)\frac{1}{T}\sum_{t=1}^{T}(x_{it-1}e_{it})+\frac{1}{T}\sum_{t=1}^{T}(\varepsilon_{it}e_{it}) \\
\Rightarrow \quad & (1-\rho)\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int W_i dV_i\right)+\varpi_{\varepsilon e}\left(\int W_i dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e},
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i=(1-\rho)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\Rightarrow(1-\rho)\,\varpi_e V_i\,(1)\,,$$

as $T\to\infty$. Hence,

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\{(1-\rho)\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int W_i dV_i\right)+\varpi_{\varepsilon e}\left(\int W_i dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e} \\
& -\frac{1}{(1-\rho)^2}(1-\rho)^3\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int W_i\right)V_i\,(1)+\varpi_{\varepsilon e}\left(\int W_i\right)W_i\,(1)\right]\}+o_p(1) \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\{(1-\rho)\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int \tilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int_i \tilde{W} dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\} \\
& \xrightarrow{p} \frac{1}{\varpi_e^2}\left[(1-\rho)\left[-\frac{1}{2}\varpi_{\varepsilon e}+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right]
\end{aligned}$$

as $n\to\infty$.

And because

$$\begin{aligned}
\frac{1}{n\sqrt{T}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i+\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\\
&= \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\right)\mu_i\right]+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]\\
&\Rightarrow o_p(1)+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{(1-\rho)\varpi_e}{\sigma_\mu^2}V_i(1)\right]\\
&\xrightarrow{p} 0
\end{aligned}$$

and also because $\frac{1}{n\sqrt{T}}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n\sqrt{T}}\\
&\xrightarrow{p}\frac{1}{\varpi_e^2}\left[(1-\rho)\left[-\frac{1}{2}\varpi_{\varepsilon e}+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right]
\end{aligned}$$

Note that

$$\frac{1}{\sqrt{n}T}G_2-\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)\,\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right]$$

First consider that

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\mathbf{x}'\Phi^{-1}\mathbf{u}\\
=\;&\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]
\end{aligned}$$

For a fixed n, because $\frac{1}{T\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\Rightarrow(1-\rho)^2\varpi_\varepsilon\int W_i+o_p(1)$, which is proved in (a) and

$$
\begin{aligned}
& \frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i \\
= \quad & \frac{1}{T^2}\left[(1-\rho)\sum_{t=1}^{T}(x_{it}-\rho x_{it-1})\right]\left[(1-\rho)\sum_{t=1}^{T}e_{it}\right] \\
= \quad & (1-\rho)^2\frac{1}{T^2}\left[\sum_{t=1}^{T}((1-\rho)x_{it-1}+\varepsilon_{it})\right]\left[\sum_{t=1}^{T}e_{it}\right] \\
= \quad & (1-\rho)^3\left[\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}x_{it-1}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right]+(1-\rho)^2\frac{1}{\sqrt{T}}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right]\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}\right] \\
\Rightarrow \quad & (1-\rho)^3\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int W_i\right)V_i(1)+\varpi_{\varepsilon e}\left(\int W_i\right)W_i(1)\right]+\frac{1}{\sqrt{T}}o_p(1)
\end{aligned}
$$

as $T\to\infty$. Hence,

$$
\begin{aligned}
& \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[(1-\rho)\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int_i\tilde{W}dV_i\right)+\varpi_{\varepsilon e}\left(\int_i\tilde{W}dW_i\right)\right]+\sigma_{\varepsilon e}\right] \\
& -\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\varpi_{\varepsilon e}\left(\int_i\tilde{W}dW_i\right)+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
= \quad & \frac{1-\rho}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int_i\tilde{W}dV_i\right)\right] \\
\Rightarrow \quad & N\left(0,\frac{(1-\rho)^2\varpi_{\varepsilon}^2\varpi_{e.\varepsilon}}{6\varpi_e^4}\right)
\end{aligned}
$$

as $n\to\infty$.

And because

$$
\begin{aligned}
\frac{1}{\sqrt{nT}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \quad &= \quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i+\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
&= \quad \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\mu_i+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} \\
&\Rightarrow \quad o_p(1)+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{(1-\rho)\varpi_e}{\sigma_\mu^2}V_i(1)\right] \\
&\Rightarrow \quad \frac{(1-\rho)\varpi_e}{\sigma_\mu^2}N(0,1)
\end{aligned}
$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}'_{NT}\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}G_2 - \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)\,\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
= \;&\frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)\,\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
&-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u}}{\sqrt{nT}} \\
\Rightarrow \;& N\left(0,\frac{(1-\rho)^2\,\varpi_\varepsilon^2\varpi_{e.\varepsilon}}{6\varpi_e^4}\right).
\end{aligned}$$

Note that

$$\frac{1}{nT}G_1 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}}{n}$$

First consider that

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= \;&\frac{1}{\sigma_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i - \frac{\sigma_\mu^2}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}'_T\mathbf{A}^{-1}\mathbf{x}_i}{\sqrt{T}}\right)
\end{aligned}$$

For a fixed n, because

$$\begin{aligned}
&\frac{1}{T}\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i \\
= \;&\frac{1}{T}\sum_{t=1}^{T}(x_{it}-x_{it-1})^2 \\
= \;&\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2 \\
\Rightarrow \;&\sigma_\varepsilon^2,
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0,$$

as $T \to \infty$. Hence,

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= \ & \frac{1}{\sigma_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{\sigma_\mu^2}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{\sqrt{T}}\right) \\
&\xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2}.
\end{aligned}$$

as $n \to \infty$.

Also because

$$\begin{aligned}
\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) \\
&\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}x_{i1} \\
&\xrightarrow{p} 0
\end{aligned}$$

because $\theta = 1$. And

$$\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_1 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n} \\
&\xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2}
\end{aligned}$$

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n}$$

First consider that

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \ & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{1}{\sqrt{T}}\frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right]
\end{aligned}$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0$, which is proved in part (a) and

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}e_{it} \xrightarrow{p} \sigma_{\varepsilon e}$$

and

$$\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}x_{i1}\nu_{i1}' \xrightarrow{p} 0,$$

as $T \to \infty$. Hence,

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{1}{\sqrt{T}}\frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right] \\
& \xrightarrow{p} \frac{\sigma_{\varepsilon e}}{\varpi_e^2}
\end{aligned}$$

as $n \to \infty$.

And because

$$\begin{aligned}
& \frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\
= & \frac{1}{n}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i + \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}\left(\mu_i + \nu_{i1}'\right) \\
& \xrightarrow{p} 0
\end{aligned}$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n} \\
& \xrightarrow{p} \frac{\sigma_{\varepsilon e}}{\varpi_e^2}
\end{aligned}$$

Note that

$$\frac{1}{\sqrt{nT}}G_2 = \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\sqrt{T}}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{\sqrt{n}}$$

First consider that

$$\begin{aligned}
& \sqrt{nT}\left(\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\sigma_{\varepsilon e}}{\varpi_e^2}\right) \\
= & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right] - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2}
\end{aligned}$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0$, which is proved in part (a) and

$$\begin{aligned}
& \sqrt{T}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \sigma_{\varepsilon e}\right) \\
= & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)e_{it} + o(1) - \sqrt{T}\sigma_{\varepsilon e} \\
= & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it} + o(1) - \sqrt{T}\sigma_{\varepsilon e} \\
\Rightarrow & N\left(0, \varpi_e^2\varpi_\varepsilon^2\right),
\end{aligned}$$

and $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}x_{i1}\nu_{i1}' \xrightarrow{p} 0$, which is proved in part (a), as $T \to \infty$. Hence,

$$\begin{aligned}
& \sqrt{nT}\left(\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\sigma_{\varepsilon e}}{\varpi_e^2}\right) \\
= & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right] - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2} \\
\Rightarrow & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[o_p(1) + N\left(0, \varpi_e^2\varpi_\varepsilon^2\right) - o_p(1)\right] \\
\Rightarrow & \frac{1}{\varpi_e^2}N\left(0, \varpi_e^2\varpi_\varepsilon^2\right)
\end{aligned}$$

as $n \to \infty$.

Also because

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i + \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
\Rightarrow & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}\left(\mu_i + \nu_{i1}'\right) \\
\Rightarrow & \frac{\varpi_e+\sigma_\mu}{\sigma_\mu^2}N(0,1)
\end{aligned}$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2} \\
= & \left[\frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2}\right] \\
& -\frac{1}{\sqrt{T}}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{\sqrt{n}} \\
\Rightarrow & \frac{1}{\varpi_e^2} N\left(0, \varpi_e^2\varpi_\varepsilon^2\right)
\end{aligned}$$

■

Proof of Theorem 4:

Proof. By $\widehat{\beta}_{GLS} - \beta = G_1^{-1}G_2 = \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{x}\right]^{-1}$ $\left[\mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{u}\right]$, the proof of Theorem 4 is straightforward with above lemmas. ■